Compactness Theorem

Recap

- For a set of formulas Γ and valuation v
  $v \models \Gamma$ iff $\forall \varphi \in \Gamma$ $v \models \varphi$
- $\Gamma \models \varphi$ iff $\forall v.$ if $v \models \Gamma$ then $v \models \varphi$
- φ is a tautology / valid if $\emptyset \models \varphi$
  or $\forall v.$ $v \models \varphi$
- φ / Γ is satisfiable if there is a valuation
  v s.t. $v \models \varphi$ ($v \models \Gamma$).

Validity Problem Given a formula φ, determine if φ is valid / tautology

Satisfiability Problem Given φ, determine if φ is satisfiable

Is $p \to (q \to p)$ (p, q propositions) valid?

$\cancel{p}$ φ	$\cancel{q}$ ψ	$\cancel{p \to (q \to p)}$ $\varphi \to (\psi \to \varphi)$	
F	F	T	T
F	T	T	F
T	F	T	T
T	T	T	T

If $\varphi = p \vee \neg p$ then this row corresponds to empty set.

Algorithm for Satisfiability and Validity
Construct the truth table

Analysis: # rows in 2^n where n is

size of the formula

p	q	$p \to q$	$(p \to q) \to p$
F	F	T	F
F	T	T	F
T	F	F	T
T	T	T	T

$(p \to q) \to p$ satisfiable

Proposition φ is valid iff $\neg\varphi$ is unsatisfiable

Compactness Theorem

Finitely Satisfiable A set of formulas Γ is finitely satisfiable if every finite subset of Γ is satisfiable

Compactness Theorem Γ is satisfiable iff Γ is finitely satisfiable.

($\Rightarrow$) If Γ satisfiable $\Rightarrow \exists v \; v \models \Gamma$

For any finite subset $\Gamma_0 \subseteq \Gamma \quad v \models \Gamma_0$

Γ is finitely satisfiable.

($\Leftarrow$) Γ is finitely satisfiable

Propositions can be enumerated as

$p_1, p_2 \ldots p_n \ldots$

Proposition: Let Δ be any set of formulas and φ be formula. If Δ is finitely satisfiable then either $\Delta \cup \{\varphi\}$ is finitely satisfiable or $\Delta \cup \{\neg\varphi\}$ is finitely satisfiable.

Proof Assume that $\Delta \cup \{\varphi\}$ and $\Delta \cup \{\neg\varphi\}$ are both <u>not</u> finitely sat.

$\exists$ finite subset $\Delta_0 \subseteq \Delta \cup \{\varphi\}$ and $\Delta_1 \subseteq \{\neg\varphi\} \cup \Delta$ s.t.
Δ_0, Δ_1 are unsatisfiable.

$\Gamma = (\Delta_0 \cup \Delta_1) \setminus \{\varphi, \neg\varphi\}$

- Γ is finite set
- $\Gamma \subseteq \Delta$
- Γ is satisfiable

$\exists$ valuation v s.t. $v \models \Gamma$

Either $v[\![\varphi]\!] = T \Rightarrow v \models \Gamma \cup \{\varphi\}$
$\Rightarrow v \models \Delta_0$ (because $\Delta_0 \subseteq \Gamma \cup \{\varphi\}$)

Or $v[\![\varphi]\!] = F \Rightarrow v \models \Gamma \cup \{\neg\varphi\}$
$\Rightarrow v \models \Delta_1$ ($\Delta_1 \subseteq \Gamma \cup \{\neg\varphi\}$)

Contradiction.

Assume Γ is finitely satisfiable

Enumerate $p_1, p_2, p_3 \ldots$

$\Delta_0 = \Gamma$

$\Delta_{i+1} = \begin{cases} \Delta_i \cup \{p_i\} & \text{if } \Delta_i \cup \{p_i\} \text{ is finitely sat} \end{cases}$

$\sim \Delta_i \cup \{\neg p_i\}$ o.w.

- $\forall i$ Δ_i is finitely satisfiable
  Prove by induction

Define $\Delta = \bigcup_{i \in \mathbb{N}} \Delta_i$

- Δ is finitely satisfiable
  Suppose Δ' is a finite subset of Δ
  $\forall i, j$ $i < j$, $\Delta_i \subseteq \Delta_j$
  $\exists i$ $\Delta' \subseteq \Delta_i \Rightarrow \Delta'$ satisfiable

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$\Delta_0 = \Gamma$ $\quad \Delta_1 = \begin{cases} \Gamma \cup \{p_1\} \\ \Gamma \cup \{\neg p_1\} \end{cases}$

$\Delta_2 = \begin{cases} \Delta_1 \cup \{p_2\} \\ \Delta_1 \cup \{\neg p_2\} \end{cases}$

Define $v(p) = \begin{cases} T & \text{if } p \in \Delta \\ F & \text{if } \neg p \in \Delta \end{cases}$

<u>Claim</u>: $v \models \Gamma$

<u>Proof</u>: Consider $\varphi \in \Gamma$ (Want to show $v \models \varphi$)

Let P = set propositions that appear in φ

$P^{\neg} = \{\neg p \mid p \in P\}$

$\Gamma_0 = \{\varphi\} \cup (P \cap \Delta) \cup (P^{\neg} \cap \Delta)$

Γ_0 is finite subset of Δ

$\Rightarrow \Gamma_0$ is satisfiable $\Rightarrow v' \models \Gamma_0$

$\forall q \in P$ $\quad v'(q) = T$ iff $q \in \Gamma_0$

$v'(q) = F$ iff $\neg q \in \Gamma_0$

v and v' agree on P

By relevance lemma, $v \models \varphi$

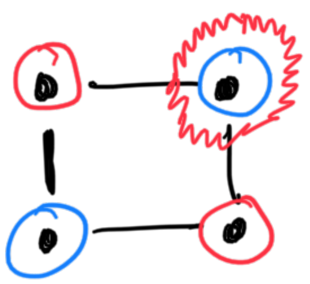
Graph (undirected simple) $G = (V, E)$

$E \subseteq V \times V$ s.t E is irreflexive $(\forall v. (v,v) \notin E)$

and E is symmetric $(\forall u, v. \ (u,v) \in E$ iff $(v,u) \in E)$

k-Coloring of a graph $G = (V, E)$ is $c : V \to \{1, \ldots k\}$

s.t $\forall u, v. \ (u,v) \in E \Rightarrow c(u) \neq c(v)$



Planar Graph is graph drawn on paper such that no two edges cross.

4-color Theorem Every finite planar graph can be colored using 4 colors.

Proposition For any graph G (finite / infinite) there is a set of formulas $\Gamma_{G,k}$ s.t.

$\Gamma_{G,k}$ is satisfiable iff G can be colored using k-colors.

Proof Take r_{ui} to denote "vertex u gets color i"

$\Gamma_{G,k}$ to be the set of formulas

- For every vertex u,

$r_{u1} \vee r_{u2} \vee \dots r_{uk}$

- For every vertex u and colors i, j

  $\neg r_{ui} \vee \neg r_{uj}$

- For every edge $(u,v) \in E$, and $\forall i$

  $\neg r_{ui} \vee \neg r_{vi}$

Proposition Every planar (infinite) graph can be colored using 4 colors.

Proof: G planar graph

$\Gamma_{G,4}$ is finitely satisfiable

$\Gamma_{G,4}$ is satisfiable