Propositional logic

Administrivia

Website courses.engr.illinois.edu/cs474/fa2021

Lecture Notes Posted on the course website

Grading Scheme 5 homeworks (every 2 weeks)
1 final exam
50% homework
50% Exam

Course Overview

Computational Problem Given input, determine if the input belongs to set (problem)

- Algorithms
- Proving lower bounds

Describing a problem in logic gives you algorithms and provides lower bounds

Course Structure

- One month (1/3) on propositional logic
- One month (1/3) on first order logic and decidable theories
- One month (1/3) on monadic second order logic, automata theory, graph algorithms, descriptive complexity

Proposition Statement that is true or false

Champaign is the capital of Illinois

What is this? Go there!

Well formed Formulas is a smallest set of strings over $Prop = \{p_1, p_2 \dots\}$, (,), $\neg$, $\vee$ that satisfies the following properties.

1. Any proposition $p_i \in Prop$ is wff
2. If φ is wff then $(\neg\varphi)$ is wff
3. If φ and ψ are wff then $(\varphi \vee \psi)$ is wff

syntactic category

$$\varphi ::= p \mid (\neg\varphi) \mid (\varphi \vee \varphi)$$

$p \in Prop$

BNF

Examples $(\neg((\neg p_1) \vee (\neg p_2)))$

$p_1 \vee$, $(p_1 (\vee$

$$wff = \{ p_1, p_2 \dots, (\neg p_1), (\neg p_2), \dots (p_1 \vee p_2), \dots (\neg(\neg p_1)), ((\neg p_1) \vee (p_1 \vee p_2)), \dots \}$$

All strings $= \{ p_1, p_2 \dots, (,),)(, p_1\vee, \dots \}$

$(\neg()$

Other operators

$\varphi \wedge \psi = (\neg((\neg\varphi) \vee (\neg\psi)))$ and / conjunction

$$\varphi \to \psi = ((\neg\varphi) \vee \psi) \qquad \text{implication}$$
$$\top = (\varphi \vee (\neg\varphi)) \qquad \text{true}$$
$$\bot = \neg\top \qquad \text{false}$$

Simplification

- Omit outermost parentheses
- Operator precedence : $\neg, \wedge, \vee, \to$

Example $\neg p \wedge q \to r$ means
$(((\neg p) \wedge q) \to r)$

Semantics

Truth Assignment / Valuation $v : \mathrm{Prop} \to \{T, F\}$

$v \models \varphi$ "v satisfies φ"

$v \models p$ iff $v(p) = T$

$v \models (\neg\varphi)$ iff $v \not\models \varphi$

$v \models \varphi \vee \psi$ iff $v \models \varphi$ or $v \models \psi$

Example $\varphi = \neg(\neg p \vee \neg q) \vee (\neg p \vee \neg q)$

v_1 : assigns all propositions to F

$v_1 \not\models p$ because $v(p) \neq T$

$v_1 \models (\neg p)$

$v_1 \models (\neg p \vee \neg q)$

$v_1 \models \neg(\neg p \vee \neg q) \vee \underline{(\neg p \vee \neg q)}$

$v[\![\varphi]\!]$ "value of φ under v"

$v[\![p]\!] = v(p)$

$$v[\![\neg\varphi]\!] = \begin{cases} T & \text{if } v[\![\varphi]\!] = F \\ F & \text{if } v[\![\varphi]\!] = T \end{cases}$$

$$v[\![\varphi \vee \psi]\!] = \begin{cases} F & \text{if } v[\![\varphi]\!] = F = v[\![\psi]\!] \\ T & \text{otherwise} \end{cases}$$

Example:

$$v_1[\![p]\!] = F$$
$$v_1[\![\neg p]\!] = T$$
$$v_1[\![\neg p \vee \neg q]\!] = T$$
$$v_1[\![\neg(\neg p \vee \neg q) \vee (\neg p \vee \neg q)]\!] = T$$

Relevance Lemma For any wff φ and valuations v_1 and v_2 s.t. $\forall$ propositions p occuring in φ, $v_1(p) = v_2(p)$ then

$$v_1[\![\varphi]\!] = v_2[\![\varphi]\!]$$

Proposition For any φ and v,

$$v \models \varphi \quad \text{iff} \quad v[\![\varphi]\!] = T$$

Logical Equivalence $\varphi \equiv \psi$ iff $\forall$ valuations v

$$v[\![\varphi]\!] = v[\![\psi]\!] \text{ or}$$
$$v \models \varphi \text{ iff } v \models \psi.$$

Example $\varphi = p \wedge (q \vee r)$ $\quad \psi = (p \wedge q) \vee (p \wedge r)$

$$\varphi \equiv \psi$$

p	q	r	$p \wedge (q \vee r)$		$(p \wedge q) \vee (p \wedge r)$		
F	F	F	F	F	F	F	F
F	F	T	F	T	F	F	F
F	T	F	F	T	F	F	F
F	T	T	F	T	F	F	F
T	F	F	F	F	F	F	F
T	F	T	T	T	F	T	T

T	T	F	T	T	T	T	F
T	T	T	T	T	T	T	T

$v_1(p) = F$, $v_1(q) = F$, $v_1(r) = F$ $v_1(s) = F$, $v_1(t) = F$ ..

$v_2(p) = F$, $v_2(q) = F$, $v_2(r) = F$ $v_1(s) = T$, $v_2(t) = T$ ..

Satisfiable Γ is a set of formulas and v is valuation $v \models \Gamma$ iff $\forall \varphi \in \Gamma$, $v \models \varphi$

Logical Consequence Γ is set of formulas

φ is wff.

$\Gamma \models \varphi$ iff $\forall v$, if $v \models \Gamma$ then $v \models \varphi$

Tautology / Valid φ is a tautology if

$\phi \models \varphi$

Take $v \models \phi$? T

φ is tautology if $\forall v.$ $v \models \varphi$

Satisfiable φ is satisfiable if there is some valuation v such that $v \models \varphi$.

Proposition: If φ is a tautology then for any set Γ, $\Gamma \models \varphi$

For any $\Gamma_1, \Gamma_2, \varphi$ s.t. $\Gamma_1 \subseteq \Gamma_2$

if $\Gamma_1 \models \varphi$ then $\Gamma_2 \models \varphi$