CS473 Algorithms: Lecture 8 (2022-02-10)

Logistics:
- pset 2 due F 17
- pset 3 out F 17

last lecture: flows
- def
- residual graphs
- augmenting paths
- integrality
- Ford Fulkerson
  - algo
  - complexity

today: flows

def: a *capacitated graph* is a simple directed graph $G=(V,E)$ w/ edge capacities $(c_e)_{e\in E}$ over $\mathbb{N}$. For $s,t\in V$ an (s,t)-flow on G is a sequence of edge flows $f=(f_e)_{e\in E}$ over $\mathbb{R}_{\geq 0}$

st
- capacity constraints $0\le f_e \le c_e$, $e\in E$
- conservation constraints

$$f^{in}(v) := \sum_{e:\,\cdot\xrightarrow{e} v} f_e$$

$$f^{out}(v) := \sum_{e:\,v\xrightarrow{e}\cdot} f_e$$

$$f(v) = f^{out}(v) - f^{in}(v)$$

$$\forall v\in V\setminus\{s,t\} \quad f(v)=0$$

the *value* of (s,t)-flow is $|f| := f(s)$

the *max (s,t)-flow problem* is to compute $\max_{f\ (s,t)\text{-flow}} |f|$ ∧ arg

Q: compute max flow?

<u>idea</u>: repeatedly push flow along $s \rightsquigarrow t$ paths

allowing $\boxed{\text{backwards}}$ flow

<u>def</u>: f (s,t)-flow over capacitated $G=(V,E)$

the <u>residual graph</u> G^f is a capacitated graph $G^f = (V^f, E^f)$ st

- $V^f = V$
- $E^f = \{e : f_e < c_e\}$ ← <u>forward edges</u>

  $\cup \{\overleftarrow{e} : f_e > 0\}$ ← <u>backwards edges</u>

  ↖ reverse orientation

- capacities: for $e \in E$

  if $e \in E^f$ $\quad (c^f)_e = c_e - f_e > 0$

  $\overleftarrow{e} \in E^f$ $\quad (c^f)_{\overleftarrow{e}} = f_e > 0$

<u>eg</u>: $G \quad \bullet \xrightarrow{1/3} \bullet \quad \longmapsto \quad G^f$
$$\begin{tikzcd} \bullet \arrow[r, bend left, "2"] & \bullet \arrow[l, bend left, "1"] \end{tikzcd}$$

<u>def</u>. flow f over G, residual G^f

An <u>augmenting path</u> p is a $\boxed{\text{simple}}$ $s \rightsquigarrow t$ path in G^f. The <u>value</u> is $|p| = \min_{e \in p} (c^f)_e$

<u>Augmenting f on p</u> is flow $f+p$ where for $e \in E$
$$(f+p)_e = \begin{cases} f_e & e, \overleftarrow{e} \notin p \\ f_e + |p| & e \in p \\ f_e - |p| & \overleftarrow{e} \in p \end{cases}$$

<u>prop</u>: $f+p$ is a valid w/ value $|f+p| = |f| + \underbrace{|p|}_{>0 \Leftarrow (c^f)_e > 0} > |f|$

idea: repeatedly augment flow

algo (Ford Fulkerson):

(1) $f_e \leftarrow 0 \quad \forall e \in E$

(2) init G^f

(3) while exists augmenting path p in G^f

(a) $f \leftarrow f + p$

(b) $G^f \leftarrow G^{f+p}$

(4) return f

prop: capacitated G w/ (integral) capacities

$\Rightarrow$ during FF, all flows integral

prop: (any) flow f in G has value $\leq \sum_{e \in E} c_e =: C$

cor: FF takes $O(C)$ iterations

$O(mC)$ time

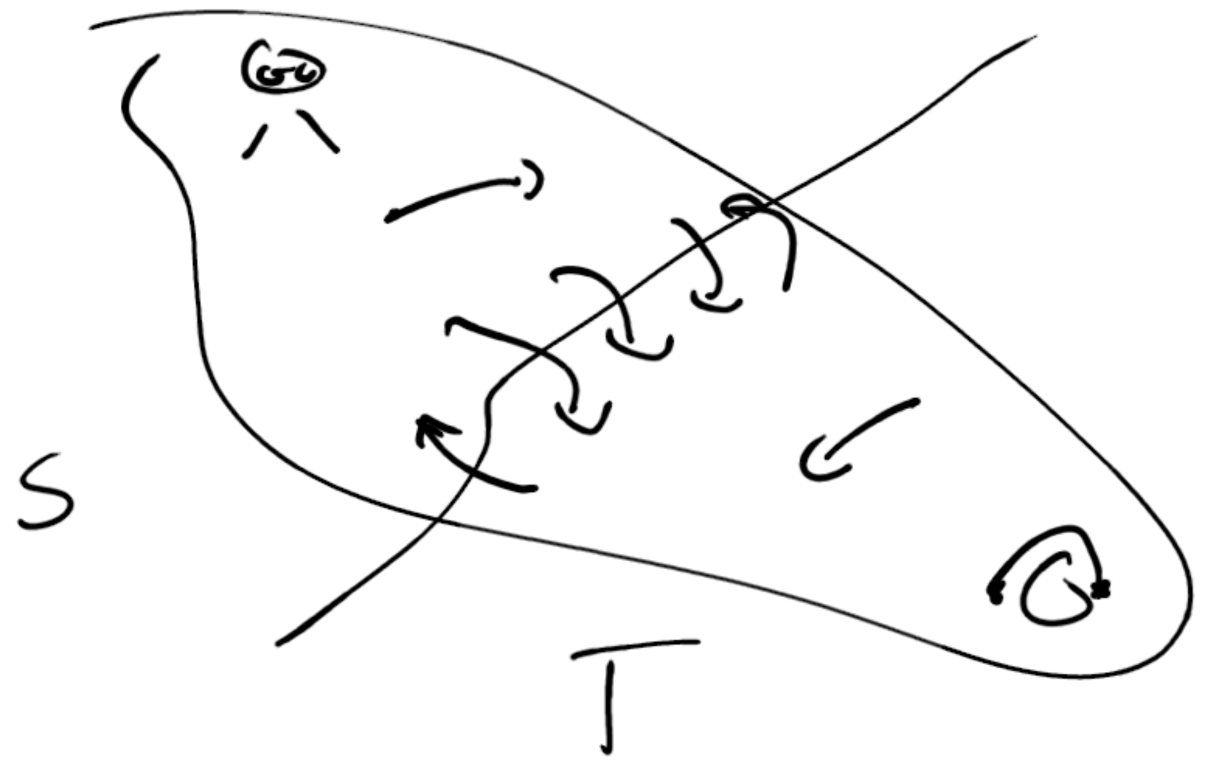
Q: correctness?

Q: why is internet slow?

def: G capacitated graph w/ $s,t\in V$.

An (s,t)-cut C is a partition $V = S \sqcup T$ w/ $s\in S$, $t\in T$. The capacity of C is
$$|C| := \sum_{\substack{e:\, u \xrightarrow{e} v \\ u\in S,\ v\in T}} c_e$$

eg:



def: f flow in G, $S\subseteq V$.

The flow through S is

$$f^{out}(S) := \sum_{\substack{e:\, u \xrightarrow{e} v \\ u\in S,\ v\in V\setminus S}} f_e$$

$$f^{in}(S) := \sum_{\substack{e:\, u \xleftarrow{e} v \\ u\in S,\ v\in V\setminus S}} f_e$$

$$f(S) := f^{out}(S) - f^{in}(S)$$

eg:
$$f(v) = f^{out}(v) - f^{in}(v) = \begin{cases} 0 & v\neq s,t \\ |f| & v=s \end{cases}$$
$$f(v) = f(\{v\})$$

prop. $f(S) = \sum_{v \in S} f(v)$

pf. $$\sum_{v \in S} f(v) = \sum_{v} \Big(\underbrace{\sum_{e : v \xrightarrow{e} \cdot} f_e}_{f^{out}(v)} - \underbrace{\sum_{e : v \xleftarrow{e} \cdot} f_e}_{f^{in}(v)} \Big)$$

$$= \underbrace{\sum_{\substack{e : v \xrightarrow{e} u \\ v \in S,\ u \in V \setminus S}} (1) \cdot f_e}_{f^{out}(S)} + \underbrace{\sum_{\substack{e : v \xleftarrow{e} u \\ v \in S,\ u \in V \setminus S}} (-1) \cdot f_e}_{-f^{in}(S)} + \underbrace{\sum_{\substack{e : v \xrightarrow{e} u \\ v \in S,\ u \in S}} \underbrace{(1 - 1)}_{0} f_e}_{0} + \underbrace{\sum_{\substack{e : v \to u \\ v \in V \setminus S,\ u \in V \setminus S}} (0)\, f_e}_{0}$$

$$= f^{out}(S) - f^{in}(S) = f(S) \quad \blacksquare$$

cor. $f(t) = -f(s) = -|f|$

pf. $$\underset{\underbrace{f^{out}(V)}_{=0} - \underbrace{f^{in}(V)}_{=0}}{\underset{\parallel}{f(V)}} \overset{\text{prop}}{=} \sum_{v \in V} f(v) \overset{\text{conservation}}{=} f(s) + f(t)$$

lem: G capacitated graph, f flow in G, C cut in G. Then $|f| \le |C|$

pf: C is $V = S \sqcup T$, $s \in S$, $t \in T$

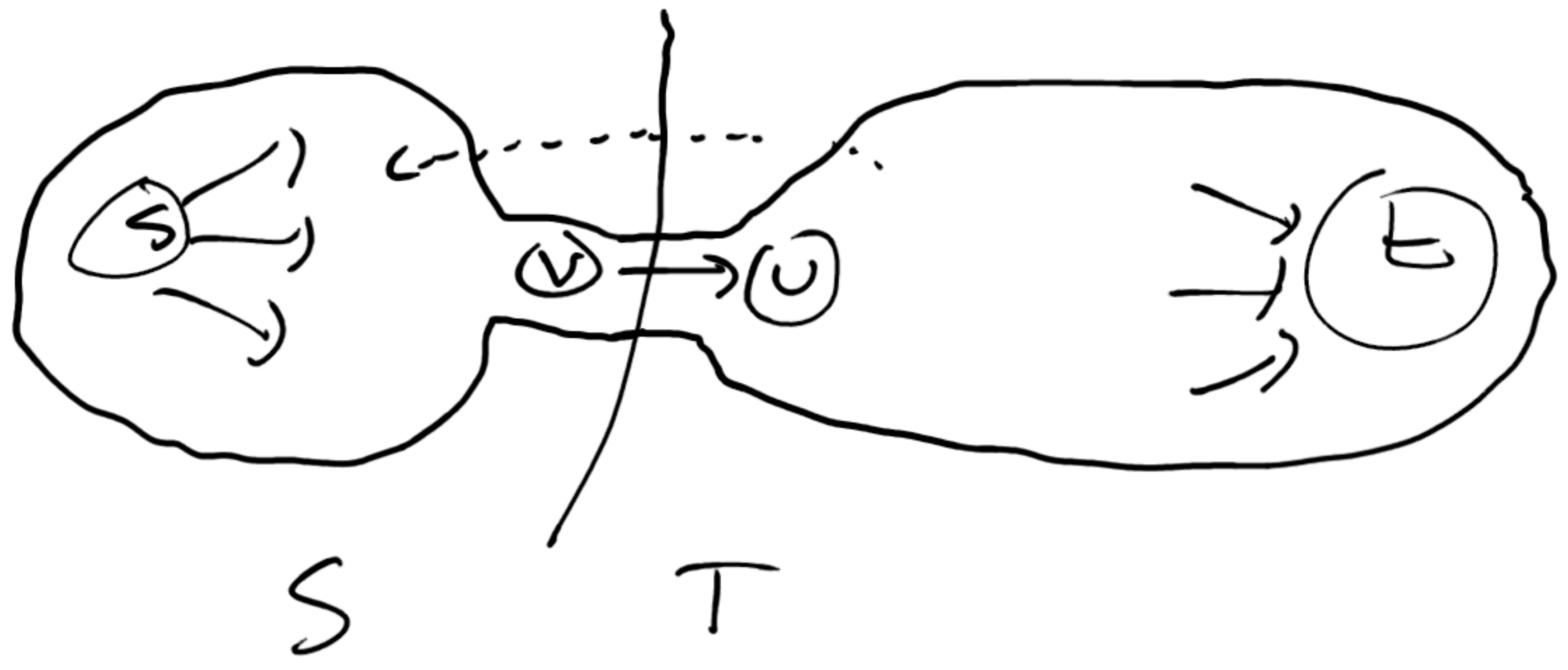
$$|f| = f(s) = f(S) = f^{out}(S) - f^{in}(S)$$
$$\le f^{out}(S)$$
$$= \sum_{\substack{e : v \xrightarrow{e} u \\ v \in S,\ u \in T}} \underbrace{f_e}_{\le c_e}$$
$$\le \sum_{e} c_e = |C| \quad \blacksquare$$

eg:

$$|f| \le |C(S,T)|$$
$$= \sum_{e : s \xrightarrow{e} \cdot} c_e$$
$$\le \underbrace{\sum_{e \in E} c_e}$$

↳ ub used to prove FF termination

eg:



$$|f| \le |C(S,T)| = c_{vu}$$

thm: $\displaystyle \max_{f\ (s,t)\text{-flows}} |f| \le \underbrace{\min_{C\ (s,t)\text{-cuts}} |C|}$

rmk: is computable by brute force

Q: FF correctness?

must implicitly provide $\boxed{\text{proof}}$ that

resulting flow is maximum

thm: max $|f| \le$ min $|C|$

prop: $\boxed{\text{if}}$ $|f| = |C| \Rightarrow$ f max flow

C min cut

idea: show FF terminates w/

prop: G capacitated graph $s, t \in V$

f flow in G. $S = \{v : s \rightsquigarrow v \text{ in } G^f\}$

equiv:

(1) G^f has no $s \rightsquigarrow t$ path

(2) $C = (S, T := V \setminus S)$ is an (s,t)-cut w/ $|C| = |f|$

(3) f max flow

pf. $\underline{\neg(1) \Rightarrow \neg(3)}$: G^f has $s \to t$ path p

$\Rightarrow$ $f+p$ flow w/ value $|f+p| = |f| + \underbrace{|p|}_{>0}$

$> |f|$

$\Rightarrow$ f not max flow

$\underline{(2) \Rightarrow (3)}$: above

$\underline{(1) \Rightarrow (2)}$: clm: C is (s,t)-cut

pf: $s \rightsquigarrow s$ is path length 0 in G^f

$\Rightarrow s \in S$

$s \not\rightsquigarrow t$ in $G^f \Rightarrow t \notin S \Rightarrow t \in T$ ☐

clm: $s \to v$ in G^f $v \xrightarrow{e} u$ in G if $f_e < c_e$

then $s \rightsquigarrow u$ in G^f

pf. $f_e < c_e \Rightarrow e \in E^f$

$\Rightarrow s \rightsquigarrow v \xrightarrow{e} u$ in G^f ☐

clm: $s \rightsquigarrow v$ in G^f $v \xleftarrow{e} u$ in G

if $\underbrace{f_e > 0}$ then $s \rightsquigarrow u$ in G^f

pf: $\Rightarrow -e \in E^f$

$\Rightarrow s \to v \to u$ in G^f ☐

cor: edge e $\underset{\in S}{v} \xrightarrow{e} \underset{\in T}{u}$ in $G \Rightarrow f_e = c_e$

pf. $\Rightarrow s \not\rightsquigarrow u$ in G^f

$s \rightsquigarrow v$ in G^f

$\Rightarrow f_e \not< c_e \Rightarrow f_e = c_e$ ☐

cor: $\underline{f^{out}(S)} = |C(S,T)|$

pf: $= \sum_{e : \underset{\in S}{v} \xrightarrow{e} \underset{\in T}{u}} \underset{= c_e}{f_e} = \underbrace{\sum_e c_e}_{|C(S,T)|}$ ☐

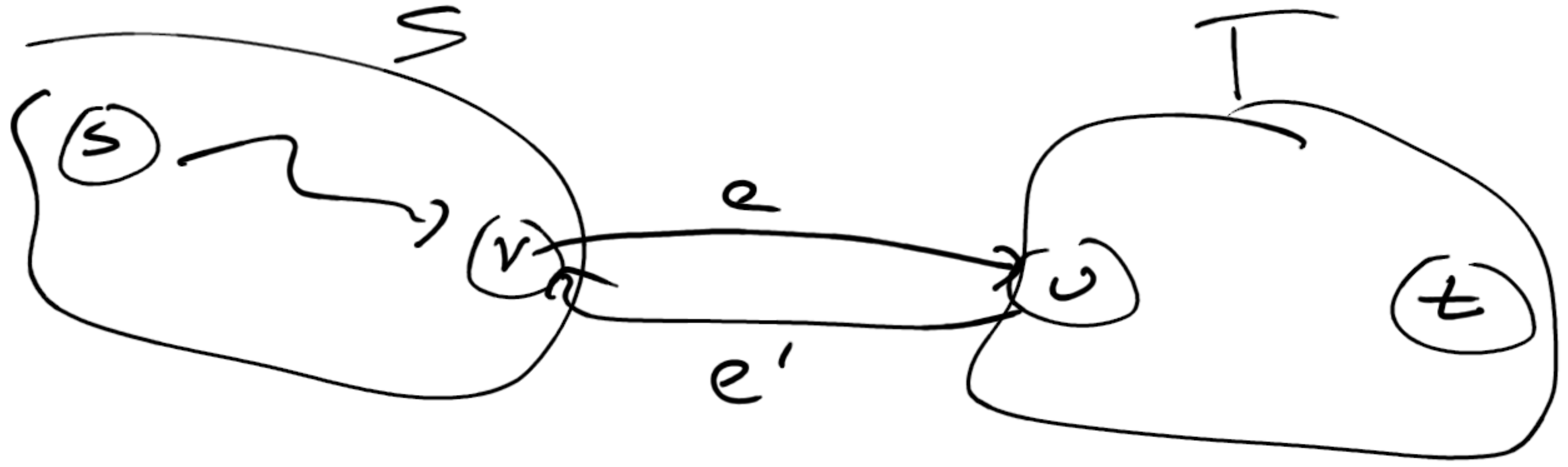
clm: edge e in G, $v \xleftarrow{e} u$, $v \in S$, $u \in T$ $\Rightarrow f_e = 0$

pf: analogous

cor: $f^{in}(S) = 0$

cor: $\underset{|f|}{f(S)} = \underbrace{f^{out}(S)}_{=|C|} - \underbrace{f^{in}(S)}_{=0} = |C|$ ∎

eg:



cor: FF terminates w/ max-flow

pf: FF terminates w/ G^f has no $s \to t$ path

$\Rightarrow f$ maxflow ∎

cor: given maxflow, can compute min cut in $O(nm)$ time

pf: take $S = \{v : s \rightsquigarrow v \text{ in } G^f\}$ ← $O(n+m)$ time via DFS

take $T = V \setminus S$

$\Rightarrow |C| = |f| \Rightarrow C$ mincut.

thm: flow values in $\boxed{\text{some}}$ max flow are integral

pf: FF produces
- max flow
- integral flow

cor: max flow can be computed via brute force

thm [max flow min cut]

G capacitated w/ $\boxed{\text{integral}}$ capacities

then $\max_{f\ (s,t)\text{ flow}} |f| = \min_{C\ (s,t)\text{-cut}} |C|$

thm [______]:

______ $\boxed{\text{real}}$ ______

Q: are we done?

today: – flow – review last lecture
– cuts
– max flow ≤ min cut
– = , if G^f has no $s \to t$ path
– = , for $\boxed{\text{integral}}$ capacities

next lecture: flows

logistics: – pset 2 due F 1/7
– pset 3 out F 1/7