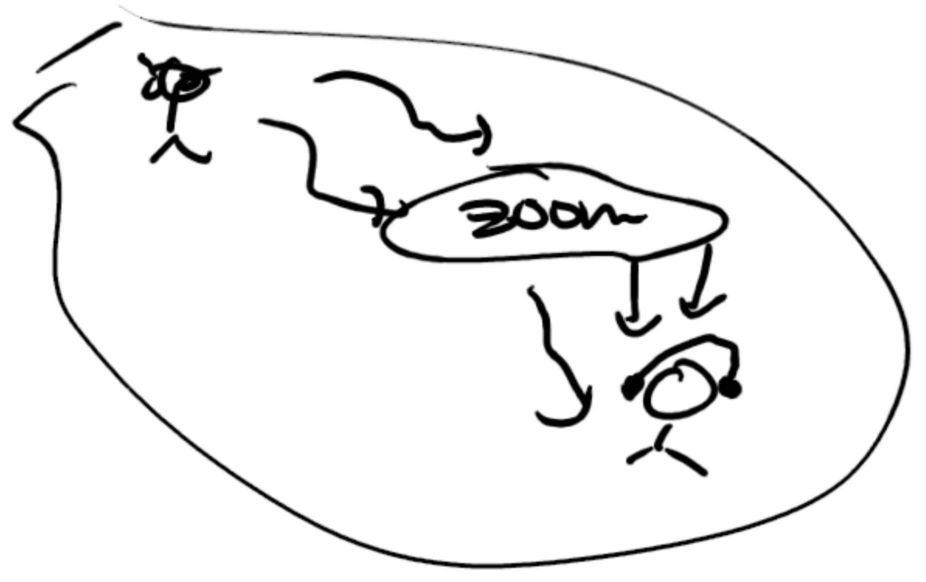
CS473 Algorithms: Lecture 7 (2022-02-08)

logistics:
- pset 2 due F17
- online this week

last lecture: dynamic programming
- Bellman Ford
  - shortest paths w/o negative cycles
  - negative cycle detection

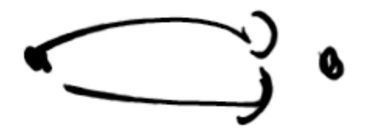
today: flows

Q: online lectures in $\boxed{\text{high definition}}$?

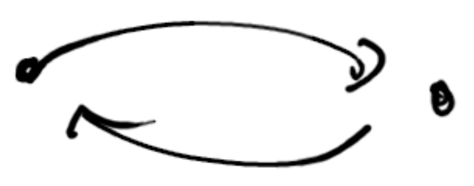


Q: how much $\boxed{\text{flow}}$ through network?

def: a directed graph $G=(V,E)$ is simple
- no isolated vertices
- no loops
- no parallel edges



note: $\boxed{\text{allow}}$ antiparallel edges



def: a capacitated graph is a simple directed graph $G=(V,E)$ w/ edge capacities $(c_e)_{e\in E}$ over $\mathbb{N}$. for $s,t\in V$ an (s,t)-flow on G is a sequence of edge flows $f=(f_e)_{e\in E}$ over $\mathbb{R}_{\geq 0}$

st:
- capacity constraints: $0\leq f_e\leq c_e,\ \forall e\in E$
- conservation:

$$f^{in}(v) := \sum_{e:\,\cdot\xrightarrow{e}v} f_e$$

$$f^{out}(v) := \sum_{e:\,v\xrightarrow{e}\cdot} f_e$$

$$f(v) := f^{out}(v) - f^{in}(v)$$

$$\forall v\in V\setminus\{s,t\}\quad f(v)=0$$

the value of (s,t)-flow is $|f| := f(s)$

the max (s,t)-flow problem is to compute

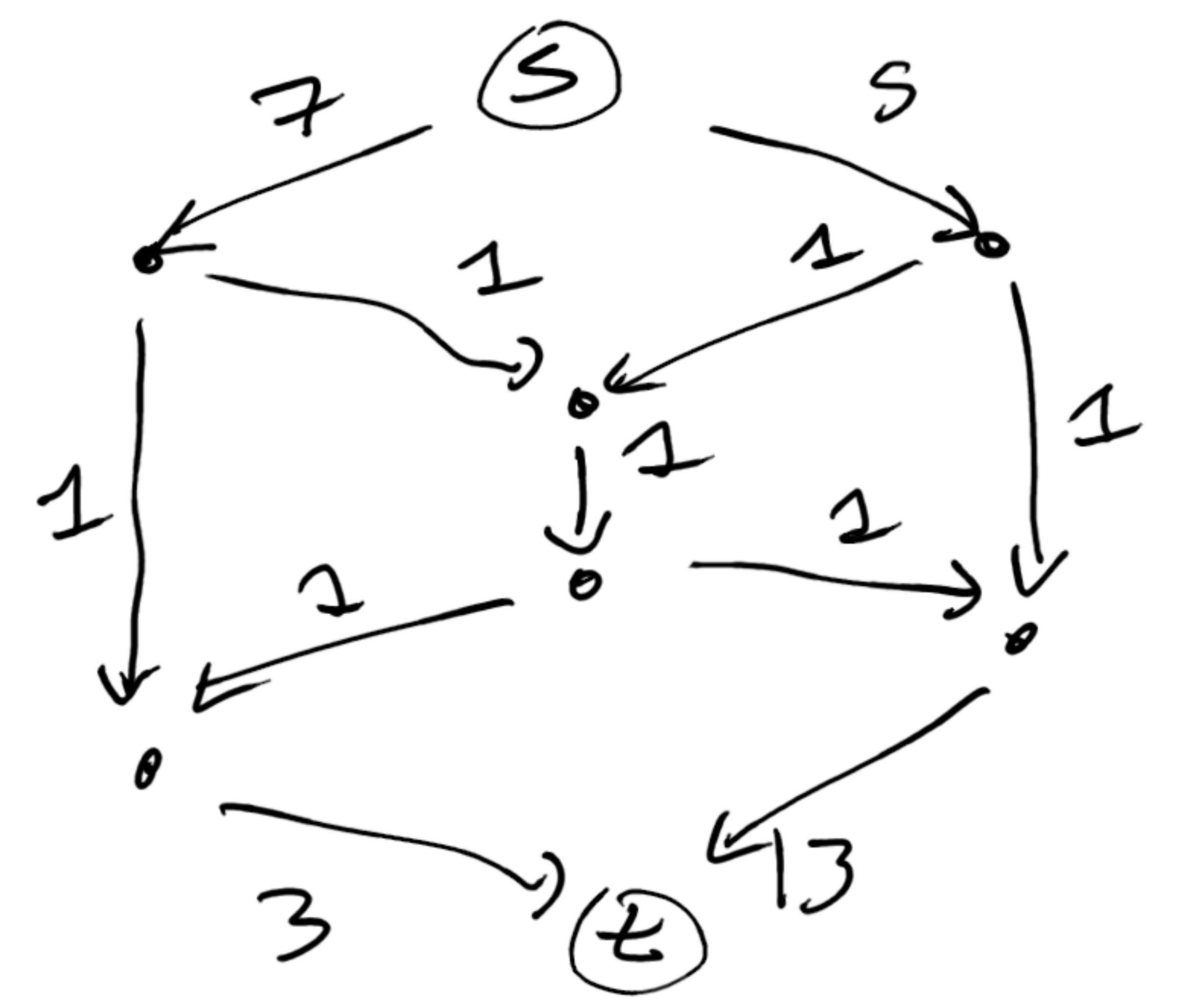
$$\underset{f\ (s,t)\text{-flow}}{\max} |f|$$

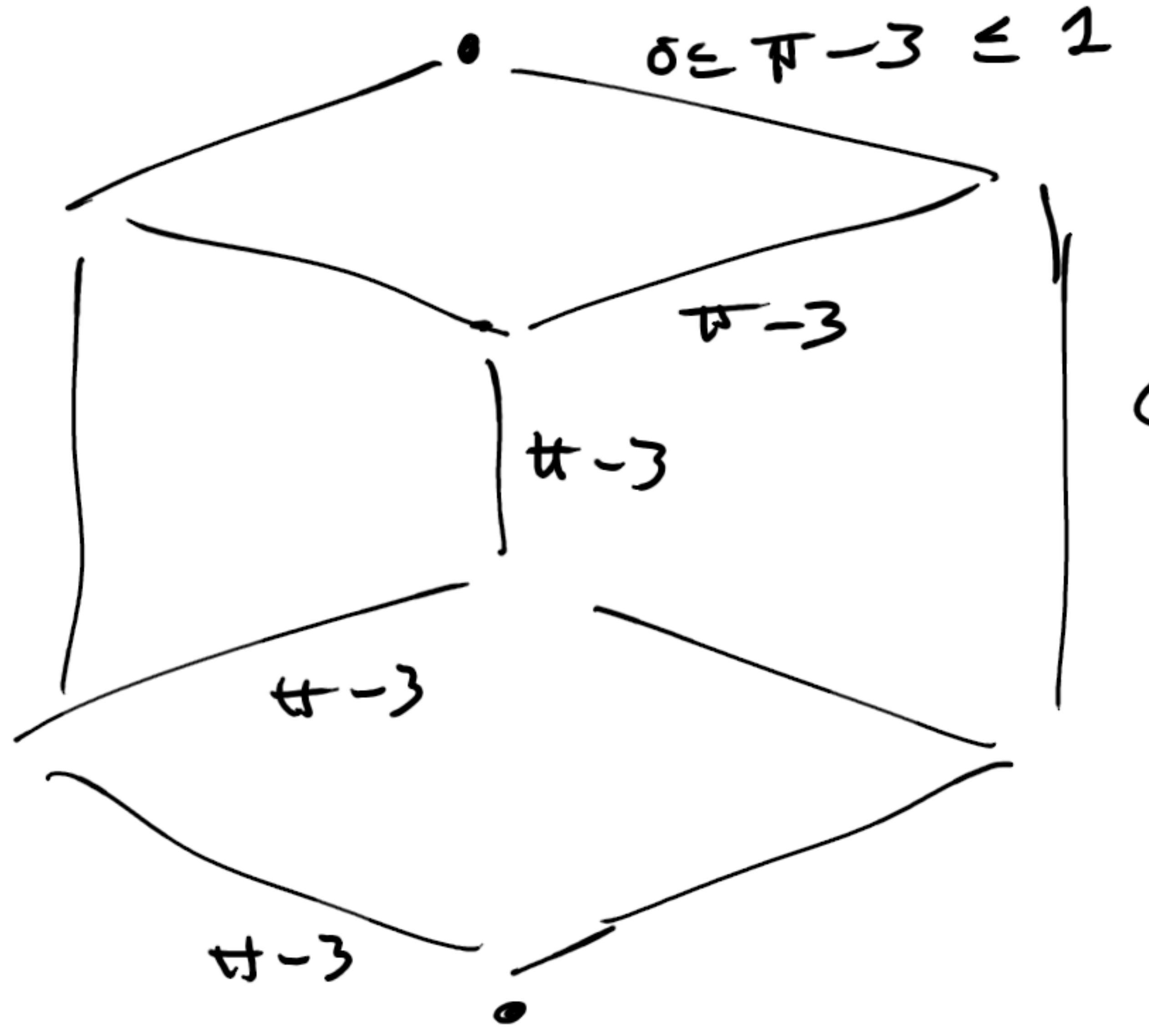
$\wedge$
arg

Q: compute max flow?

Q: compute max flow via brute force?

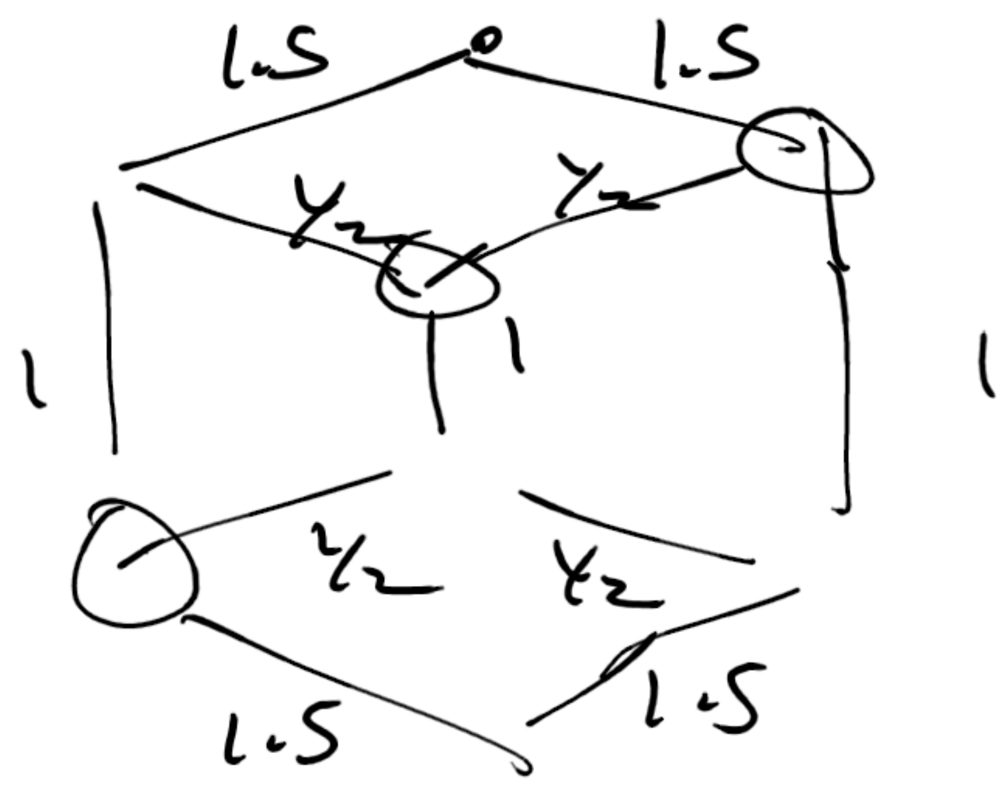
eg:





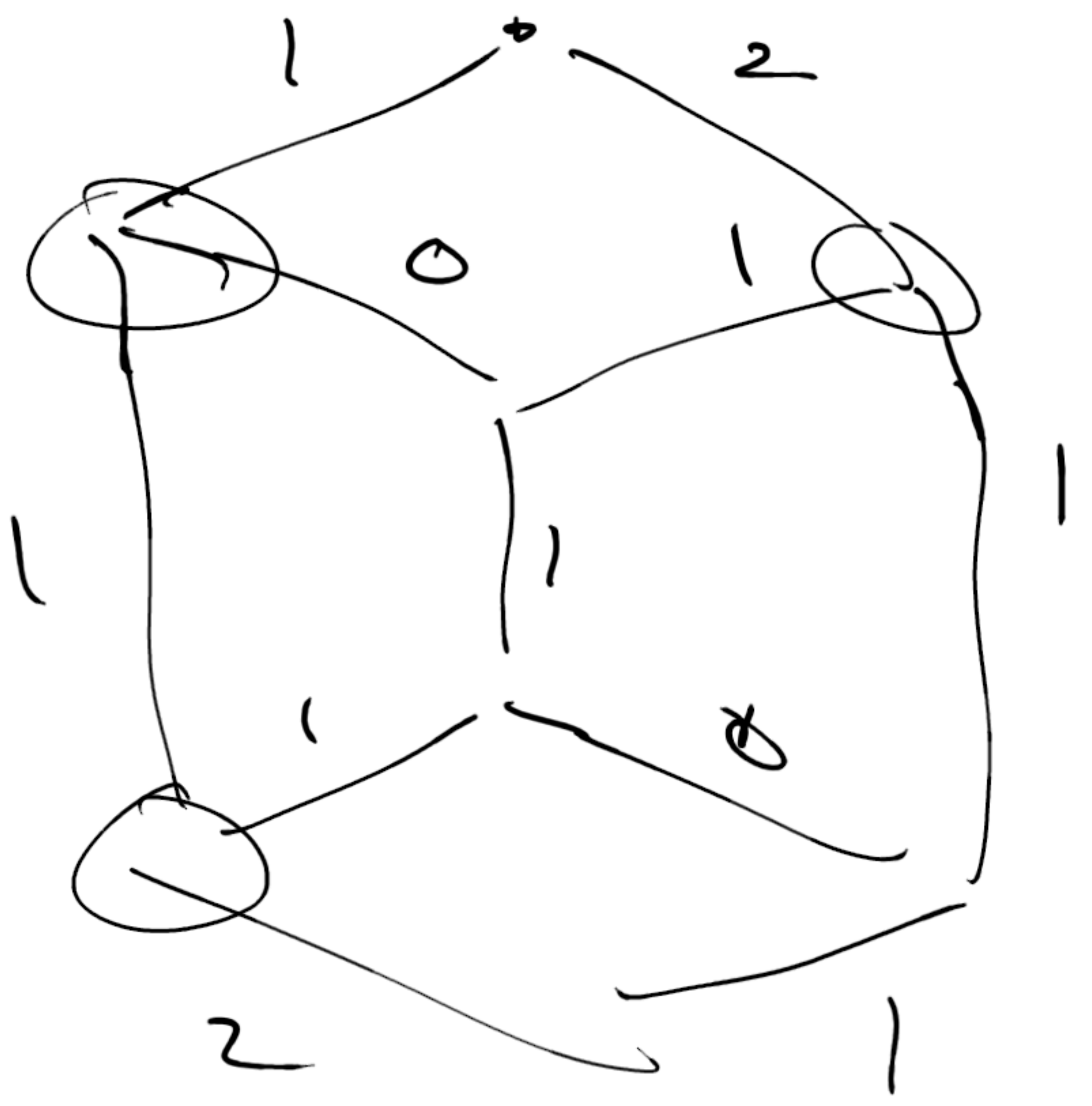
$|f| = \pi - 3 \le 1$

irrational



$|f| = 3$

rational



$|f| = 3$

integral

meta fact: integrality of optimal solution is significant issue in optimization

eg:

eg:

thm: (some) max flow is integral

cor: max flow is computable

sketch: finitely many $(f_e)_{e \in E}$ are $\mathbb{N}$ satisfying capacity constraints

↳ try them all

□

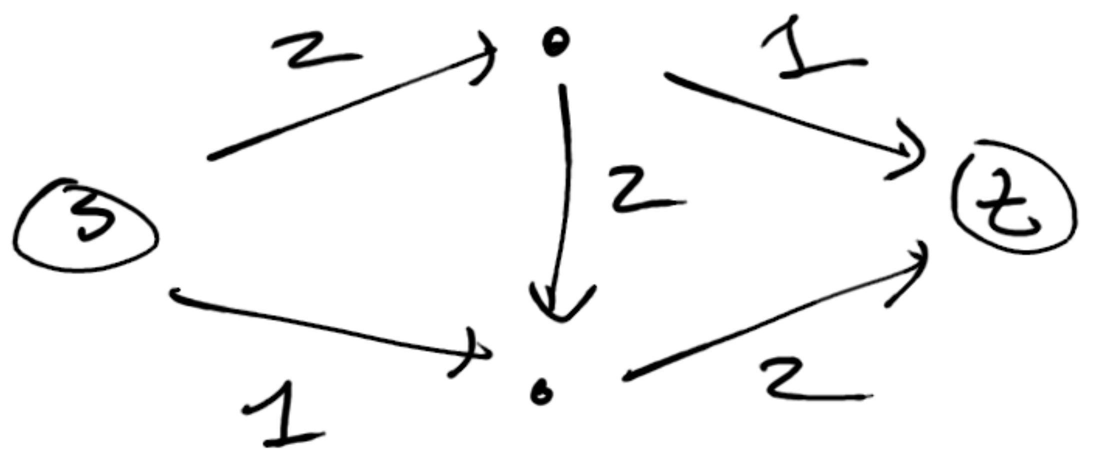
Q: compute max flow efficiently?

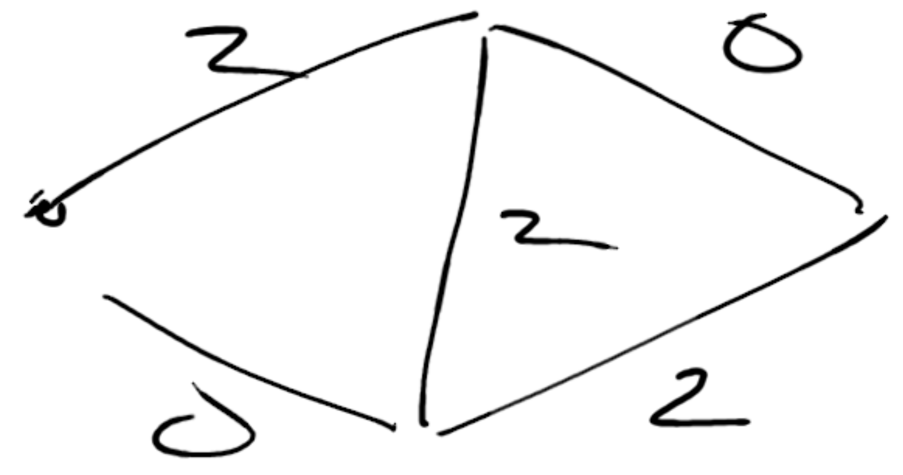
idea: dynamic programming (?)

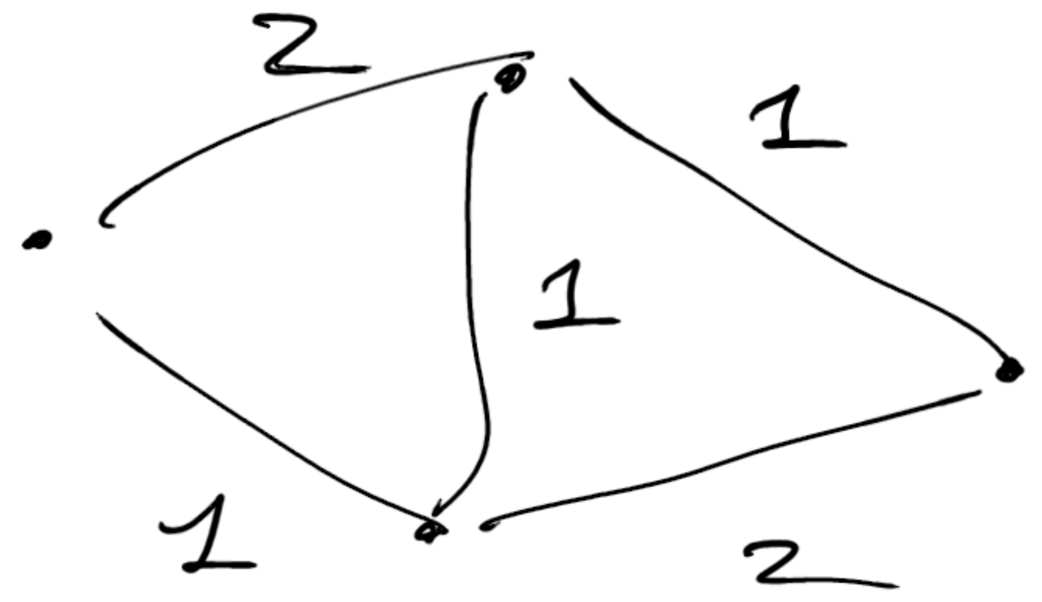
idea: greedy algo

local opt → global opt

eg:







idea: push flow along some s → t path

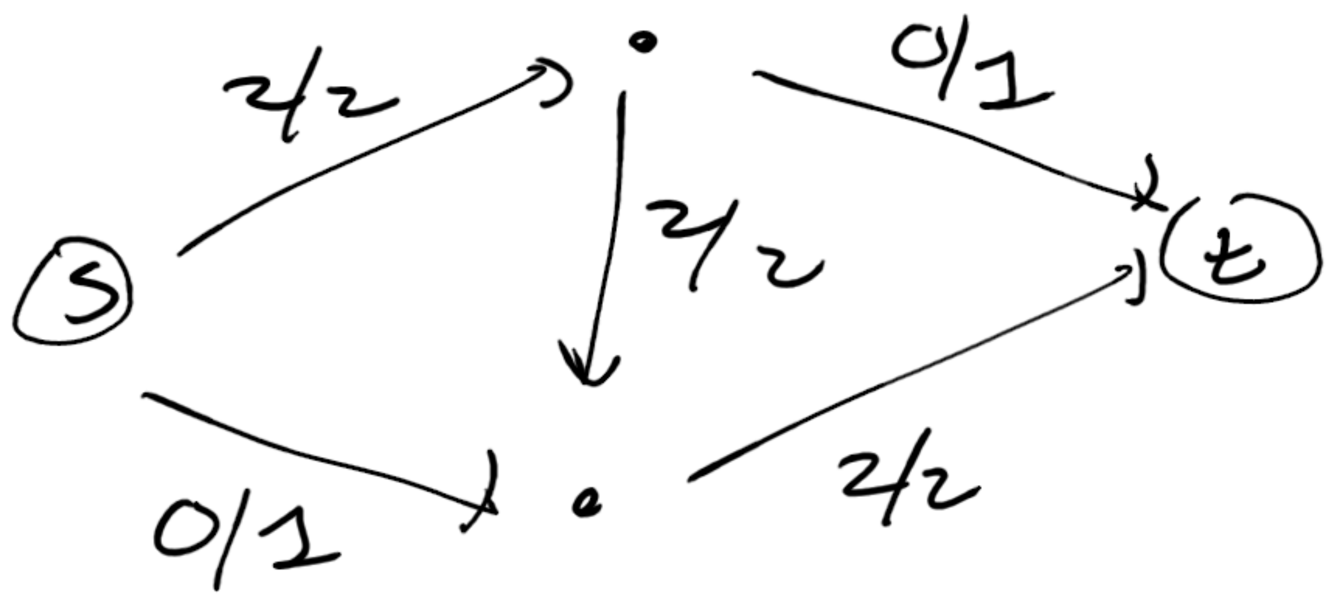
$|f| = 2$

(local) opt: no s → t path to push (more) flow

$|f'| = 3 > 2 = |f|$

⇒ f (not) global opt

idea: allow flow go backwards



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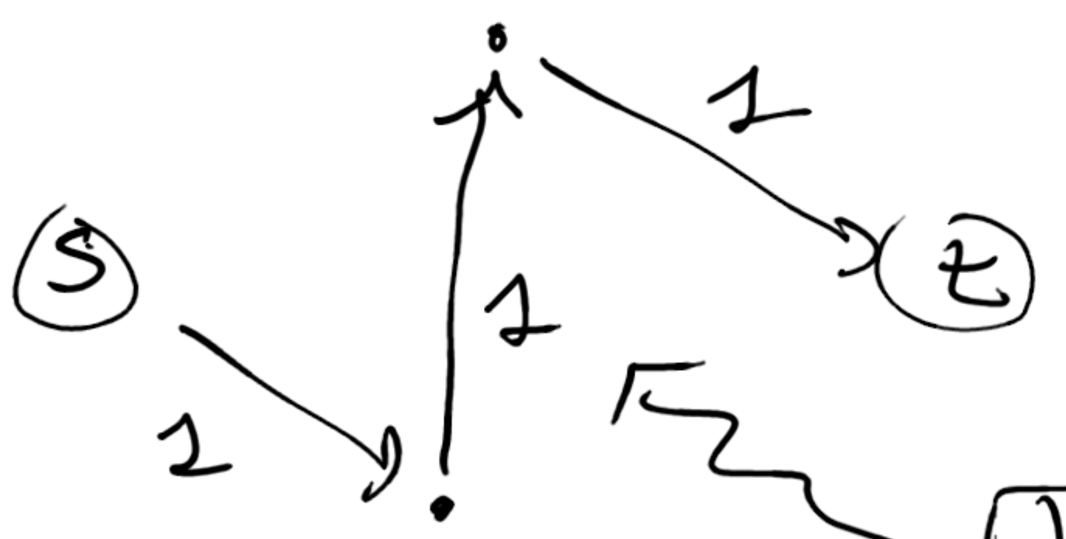


=

backwards flow

def: (s,t)-flow f over capacitated $G=(V,E)$

the <u>residual graph</u> G^f is a capacitated graph $G^f=(V^f,E^f)$ w/

- $V^f = V$
- $E^f = \{e : e\in E,\ f_e < c_e\}$

  forward edges

  $\cup \{\underbrace{-e}_{} : e\in E,\ f_e > 0\}$

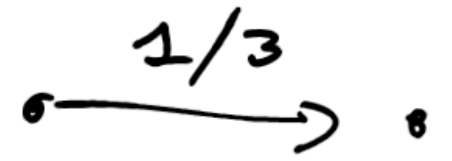
  reverse direction of e

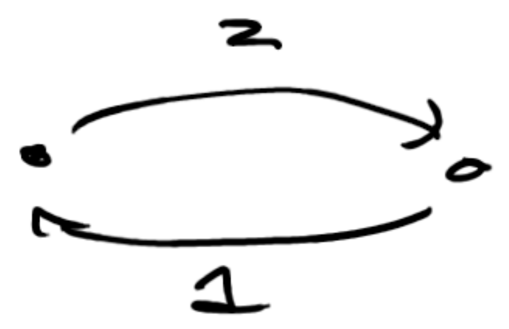
  backwards edges

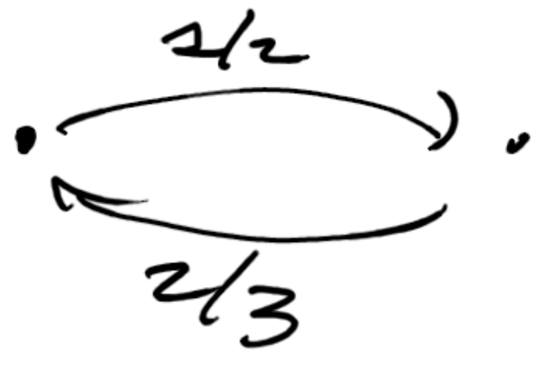
- capacities: for $e \in E$

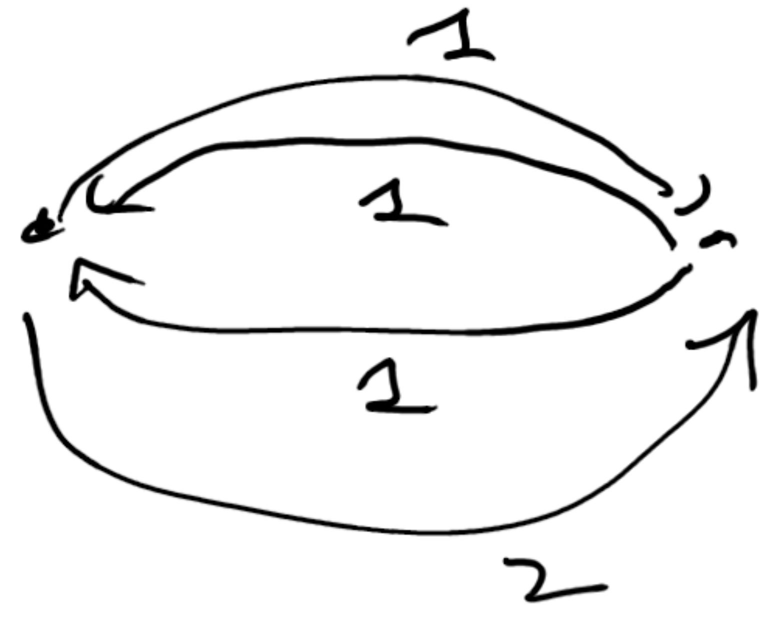
  if $e \in E^f$ $\quad (c^f)_e = c_e - f_e > 0$

  $-e \in E^f$ $\quad (c^f)_e = f_e > 0$

eg: G 

G^f 

eg: G 

G^f 

def: (s,t)-flow f on G, residual graph G^f

An <u>augmenting path</u> p is a $\boxed{\text{simple}}$ $s \to t$ path in G^f. The <u>value</u> is

$$|p| = \min_{e \in p} (c^f)_e$$

Augmenting f by p yields flow $f+p$ where $\forall e \in E$

$$(f+p)_e := \begin{cases} f_e & e, \overleftarrow{e} \notin p \\ f_e + |p| & e \in p \\ f_e - |p| & \overleftarrow{e} \in p \end{cases}$$

prop: flow f in G, p augmenting path then $f+p$ is valid flow, $|f+p| = |f| + |p|$

pf: <u>capacity</u>: $e \in E$

$\overleftarrow{e}, e \notin p$: $\quad 0 \le \underbrace{f_e}_{(f+p)_e} \le c_e$

$e \in p$: $\quad (c^f)_e = c_e - f_e \ge \min_{e' \in p} c^f_{e'} =: |p|$

$$(f+p)_e = f_e + \underbrace{|p|}_{\le c_e - f_e} \le c_e$$

$$= \underbrace{f_e}_{\ge 0} + \underbrace{|p|}_{\ge 0} \ge 0$$

$\overleftarrow{e} \in p$: $\quad (c^f)_{\overleftarrow{e}} = f_e \ge \min_{e' \in p} (c^f)_{e'} =: |p|$

$$(f+p)_e = \underbrace{f_e}_{\ge 0} - \underbrace{|p|}_{\ge 0} \le f_e \le c_e$$

$$= f_e - \underbrace{|p|}_{\le f_e} \ge 0$$

<u>Conservation</u>: <u>sketch</u>:

for $v \ne s, t$

if p visits v

4 cases of orientation

<u>eg</u>: $\xrightarrow{f_e} \bullet \xrightarrow{f_{e'}} \;\mapsto\; \xrightarrow{f_e + |p|} \bullet \xrightarrow{f_{e'} + |p|}$

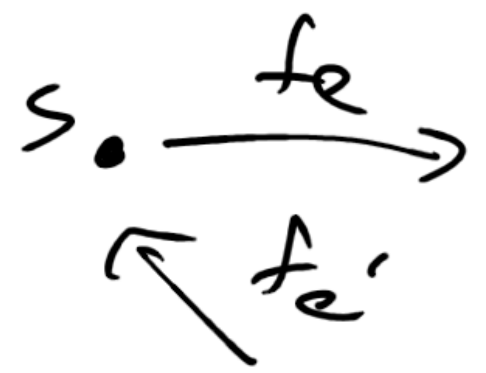
$\Rightarrow$ conservation

<u>eg</u>: $\xrightarrow{f_e} \bullet \xleftarrow{f_{e'}} \;\mapsto\; \xrightarrow{f_e + |p|} \bullet \xleftarrow{f_{e'} - |p|}$

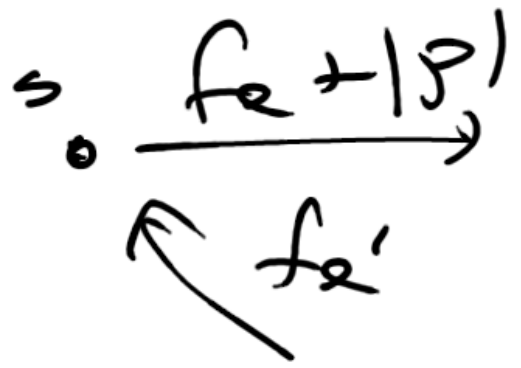
$\Rightarrow$ conservation

value: [illegible] –

p $s \rightsquigarrow t$ path



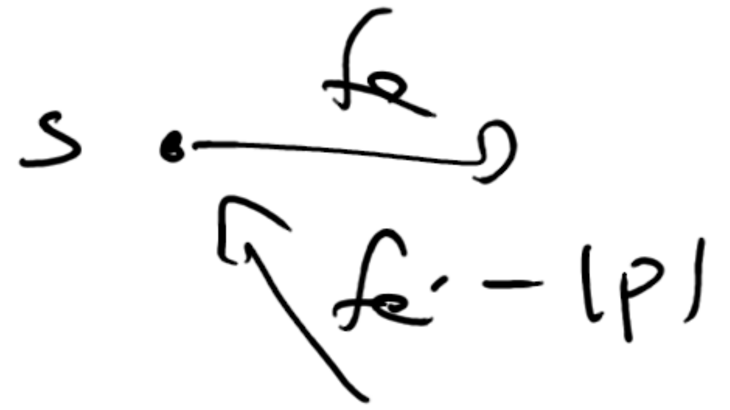
$\mapsto$ e is on p



$|f| = f(s)$

$f(s) + |p| = |f+p|$

$\mapsto$ e' is on p



$f(s) + |p| = |f+p|$

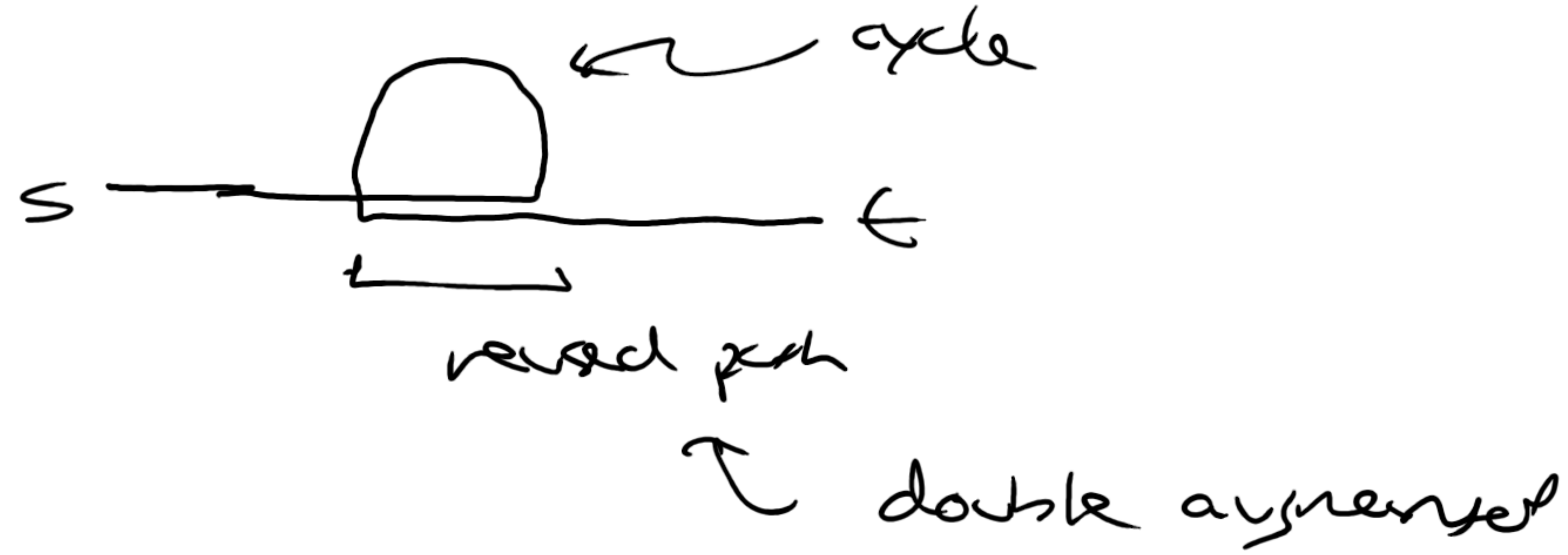
net flow

$\square$

note: $\boxed{\text{simplicity}}$ of path needed

↳ correctness of $|p|$

e.g.



idea: repeatedly augment flow

algo (Ford Fulkerson)

(1) $f_e \leftarrow 0$, $\forall e \in E$

(2) initialize G^f

(3) while exists augmenting path p in G^f

  (a) $f \leftarrow f + p$

  (b) $G^f \leftarrow G^{f+p}$

(4) return f

Q: complexity? correctness?

prop: each iteration of FF takes $O(n+m) = O(m)$ time ← no isolated vertices

sketch: clm: construction G^f from f takes $O(m)$ time

sketch: iterate over edges

- $|E^f| \le 2m$
- adjacency list

clm: finding p takes $O(n+m)$ time

sketch: depth-first search

clm: augmenting $f \mapsto f+p$ takes $O(n+m)$ time

sketch: iterate over edges, adjacency list

compute $|p| = \min_{e \in p} |c^f_e|$ □

prop: (s,t)-flow f in G, residual G^f

$\boxed{\text{if}}$ capacities $(c_e)_{e \in E}$ integral

flows $(f_e)_e$ integral

then (a) G^f has integral capacities

(b) $|p|$ integral any augmenting p

(c) $f+p$ integral

sketch:

(a) $(c^f)_e = \begin{cases} c_e - f_e & \text{\_\_\_\_} \\ f_e & \text{\_\_\_\_} \end{cases}$

(b) $|p| = \min_{e \in p} c^f_e$

(c) $(f+p)_e = \begin{cases} f_e & \text{\_\_\_\_} \\ f_e + |p| & \text{\_\_\_\_} \\ f_e - |p| & \text{\_\_\_\_} \end{cases}$

cor: in FF algo on integral capacities flows are always integral, residual capacities ____

prop: ________ $|f+p| = |f| + |p| \ge |f| + 1$

prop: (any) flow in G, have

$$|f| \le \sum_{e\in E} c_e =: C$$

pf: $|f| = f(s)$

$$= f^{out}(s) - f^{in}(s)$$

$$\le f^{out}(s)$$

$$= \sum_{e: s\to \cdot} \underbrace{f_e}_{\le c_e}$$

$$\le \sum_{e: s \to \cdot} c_e$$

$$\le \quad \square$$

cor: Ford Fulkerson, on $\boxed{\text{integral}}$ capacities

has $\sum_{e\in E} c_e = C$ # of iterations

sketch: start w/ $|f|=0$
increase $|f|$ by ≥ 1 each iteration, $|f| \le C$ always $\square$

cor: ______

run in $O(mC)$ time

Q: correctness?

today: flows
- def
- residual graph
- augmenting paths
- integrality
- Ford Fulkerson complexity

next lecture: flows

logistics: - pset 2 due 11/7
- online this week