$\overset{11}{\cancel{10}} \rightarrow \underline{\geq 20}$

cs473 Algorithms · Lecture 19 (2022-03-31)

logistics = - pset 7 due F17
- pset 8 out F17

last lecture: linear programming
- example ⟦2D⟧
- def
- reduction to canonical form ⟦formal⟧ ⟦backward⟧
- LP vs - max flow ⟦clear⟧
  - mincut ⟦discrete vs continuous⟧ ⟦randomized rounding⟧

today = linear programming ⟦last lecture⟧ ⟦profit⟧ ⟦black⟧ ⟦jamie⟧

eg ⟦bubble tea shop⟧ max $6x + 8y$

st $x, y \geq 0$ ⟦money tea made per day⟧

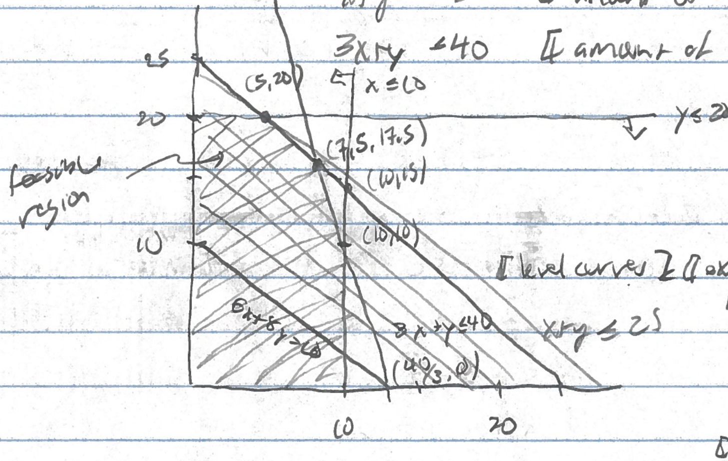
$x \leq 10$ ⟦# black tea bags⟧

$y \leq 20$ ⟦# green tea bags⟧

$x+y \leq 25$ ⟦amount of tapioca pearls⟧

$3x+y \leq 40$ ⟦amount of sugar⟧

lecture 18 ·
cross-et
reader ·
so $x,y \geq 0$
last



⟦level curves⟧ ⟦extreme points⟧

$6 \cdot 5 + 8 \cdot 20 = 190$

$6 \cdot 7.5 + 8 \cdot 17.5 = 195$ ← 2019 err 185

$6 \cdot 10 + 8 \cdot 10 = 140$

" "

$\Rightarrow$ " opt is 190

Q: how to $\boxed{\text{prove}}$ $\max_{st \ldots} \ldots = 190$? ⟦pretty picture⟧ ⟦formal/symbolic proof⟧

idea = derive $\boxed{\text{new}}$ constraints from original constraints

eg =

$x, y \geq 0$

$x \leq 10$

$y \leq 20$

$x+y \leq 25$

$3x+y \leq 40$

$$\begin{array}{r} x \leq 10 \\ + \quad y \leq 20 \\ \hline x+y \leq 30 \end{array}$$

↳ vs $x+y \leq 25$

$$\begin{array}{r} 6(x \leq 10) \equiv 6x \leq 60 \\ 8y \leq 160 \\ \hline \underbrace{6x+8y}_{\text{objective function}} \leq 220 \end{array}$$

$\Rightarrow \max_{st \ldots} 6x+8y \leq 220$

$$\begin{array}{r} 8(x+y) \leq 8 \cdot 25 \\ -2(x \geq 0) \\ \hline \underbrace{6x+8y}_{\text{objective}} \leq 200 \end{array}$$

⟦reverse inequality⟧

$\underbrace{8(x+y)}_{} \leq 8 \cdot 25 = 200$

$8x + 8y$
vs
$6x + 8y$ ⟦coordinatewise⟧ $\boxed{\text{as}}$ $x, y \geq 0$

$$\begin{array}{r} 2(3x+y) \leq 2 \cdot 40 \\ 6 \cdot y \leq 6 \cdot 20 \\ \hline \underbrace{6x+8y}_{\text{objective}} \leq 200 \end{array}$$

$$\begin{array}{r} 6(x+y) \leq 6 \cdot 25 \\ 2 \cdot y \leq 2 \cdot 20 \\ \hline 6x+8y \leq 190 \end{array}$$

$\Rightarrow \max_{st \ldots} 6x+8y \leq 190$

complementary slackness
order of constraints

$\boxed{\text{thm}}$ exist $(x,y)=(5,20)$ w/ $6x+8y=190$

$\Rightarrow$ 190 is optimum value

Q = mechanize this process? find $\boxed{\text{best}}$ ub?

$$\begin{array}{llcll}
a(& x \le 10 &) & \longmapsto & ax \le 10a \quad \boxed{\text{iff}} \quad a \ge 0 \\
b(& y \le 20 &) & \longmapsto & by \le 20b \quad \boxed{\text{iff}} \quad b \ge 0 \\
c(& x+y \le 25 &) & \dots & \qquad\qquad\qquad\quad c \ge 0 \\
d(& 3x+y \le 40 &) & & \qquad\qquad\qquad\quad d \ge 0 \\
e(& x \ge 0 &) & \longmapsto & ex \le 0 \quad \boxed{\text{iff}} \quad e \le 0 \\
+\ f(& y \ge 0 &) & \longmapsto & fy \le 0 \qquad\qquad f \le 0
\end{array}$$

[order of inequality]

[reverse]

$$\underbrace{(a+c+3d+e)}_{=6}x + \underbrace{(b+c+d+f)}_{=8}y \le 10a+20b+25c+40d+0+0$$

[dist]

$\Rightarrow$ $\geq 6x+8y$ for feasible (x,y)

Q: $\boxed{\text{best}}$ ub? [min ub]

$\boxed{\text{dual}}$ LP: min $10a+20b+25c+40d$ [min ub] [min vs max]

[LP!] st

$$a+c+3d+e=6$$
$$b+c+d+f=8$$
$$a,b,c,d \ge 0$$
$$e,f \le 0 \quad [!]$$

[slack vars]

[not canonical $- =$, $- \ge$]

[a bit better - def - correct in general]

canonical form: min $10a+20b+25c+40d$

st

$$-(a+c+3d) \le -6$$
$$-(b+c+d) \le -8$$
$$a,b,c,d \ge 0$$

[remove slack]
[flip inequalities]

clm: $\boxed{\text{any}}$ feasible (a,b,c,d) yields ub on $6x+8y$ $\forall$ (x,y) feasible

$$\Rightarrow \quad \begin{array}{ll} \max & 6x+8y \\ \text{st} & x \le 10 \\ & y \le 20 \\ & x+y \le 25 \\ & 3x+y \le 40 \\ & x,y \ge 0 \end{array} \quad \le \quad \begin{array}{ll} \min & 10a+20b+25c+40d \\ \text{st} & -(a+c+3d) \le -6 \\ & -(b+c+d) \le -8 \\ & a,b,c,d \ge 0 \end{array}$$

$\boxed{\text{and}}$ is equality

$$6 \cdot 5 + 8 \cdot 20 = 170 = 10 \cdot 0 + 20 \cdot 2 + 25 \cdot 6 + 40 \cdot 0$$

$\boxed{\text{and}}$ # var $=2$ vs 4

$\boxed{\text{non axis}}$ constraints: 4 vs 2

Q: generalize?

def: primal linear program $\Pi = \begin{array}{l} \text{⟦canonical form⟧} \\ \max \langle c,x\rangle \\ \text{st } Ax \le b \\ \quad x \ge 0 \end{array}$ $\quad c \in \mathbb{R}^n$ ⟦n vars⟧ $\quad A \in \mathbb{R}^{m \times n}$ ⟦m non axis constraints⟧ $\quad b \in \mathbb{R}^m$ ⟦n axis constraints⟧

has dual $\coprod = \min \langle b,y\rangle$ ← m vars ⟦b⟧

st $A^T y \ge c$ ← n (non axis) constraints

$y \ge 0$

matrix $\boxed{\text{transpose}}$ $(A^T)_{ij} = (A)_{ji}$ $\qquad$ convention: all vectors are $\boxed{\text{column}}$ vectors.

rmk: dual $\boxed{\text{not}}$ in canonical form, but can be made to be

$$\begin{array}{l} \max \langle c,x\rangle \\ Ax \le b \\ x \ge 0 \\ A \ m\times n \end{array} \xmapsto{\text{dual}} \begin{array}{l} \min \langle b,y\rangle \\ A^T y \ge c \\ y \ge 0 \\ A^T \ n\times m \end{array} \xleftrightarrow[\text{minus sign}]{\text{canonicalize}} \begin{array}{l} -\max \langle -b,y\rangle \\ (-A^T) y \le -c \\ y \ge 0 \\ A^T \ n\times m \end{array}$$

- many choice for "the" dual $\qquad$ isomorphic

eg:

$$\begin{array}{l} \max x \\ \text{st } x \le 1 \\ \quad x \ge 0 \end{array} \mapsto \begin{array}{l} \min y \\ y \ge 1 \\ y \ge 0 \end{array}$$

$\rotatebox{90}{$\cong$}$ $\qquad\qquad \subsetneq\ \rotatebox{90}{$\cong$}$ $\qquad$ ISO.

$$\begin{array}{l} \max x \\ \text{st } x \le 1 \\ \quad x \le 2 \\ \quad x \ge 0 \end{array} \mapsto \begin{array}{l} \min y + 2z \\ \text{st } y + z \ge 1 \\ \quad y, z \ge 0 \end{array}$$

draw mentally.

thm (weak duality): $\underset{\substack{\| \\ \max \langle c,x\rangle \\ Ax \le b \\ x \ge 0}}{|\Pi|}$ ⟦primal⟧ $\le$ $\underset{\substack{\| \\ \min \langle b,y\rangle \\ A^T y \ge c \\ y \ge 0}}{|\coprod|}$ ⟦dual⟧, $\boxed{\exists}$ both feasible

pf: x ~~feasible~~ for $\Pi \Rightarrow Ax \le b$

y —— $\coprod \Rightarrow A^T y \ge c \overset{⟦t+t⟧}{\Rightarrow} y^T A \ge c^T$ $\Bigg] \Rightarrow y^T A x$ $\overset{y \ge 0,\ Ax \le b}{\le} y^T b \overset{⟦def⟧}{=} \langle b,y\rangle$

rowvectors $\qquad\qquad\qquad\qquad \overset{x \ge 0,\ y^T A \ge c}{\ge} c^T x \overset{⟦def⟧}{=} \langle c,x\rangle$

$\Rightarrow \langle c,x\rangle \le \langle b,y\rangle$ $\boxed{\text{any}}$ x,y feasible

$\Rightarrow |\Pi| = \max\limits_{x \text{ feasible}} \langle c,x\rangle \le \min\limits_{y \text{ feasible}} \langle b,y\rangle = |\coprod|$ $\qquad$ max vs sup.

cor: - dual unbounded $\equiv |\coprod| = -\infty \Rightarrow$ primal infeasible $\qquad$ proof.

- primal unbounded $\equiv |\Pi| = \infty \Rightarrow$ dual infeasible

- $\boxed{\exists}$ x,y both feasible w/ $\langle c,x\rangle = \langle b,y\rangle \Rightarrow$ $\boxed{\text{both}}$ optimal

⟦weak vs strong⟧

thm [strong duality]: if Π feasible and bounded, then so is $\coprod$ and $|\Pi| = |\coprod|$.

· do example LP in noncanonical form

rmk: - dual(dual) = primal

- $\Longleftrightarrow$ $\amalg$ feasible, bounded $\Rightarrow$ Π is also

- strong duality $\Rightarrow$ LPs have short proofs of optimality

thm [weak duality]: $|\Pi| =$ [noncanonical] [ex of duality in more generality]

$$|\Pi| = \begin{array}{l} \max \langle c,x\rangle \\ Ax = b \\ A'x \le b' \\ x \ge 0 \end{array} \le \begin{array}{l} \min \langle b,y\rangle + \langle b',z\rangle \\ \text{st } A^T y + (A')^T z \ge c \\ z \ge 0 \\ [y \text{ unrestricted}] \end{array} = |\amalg|$$

2019 err

pf: $\langle c,x\rangle = x^T c \overset{x\ge0}{\le} x^T(A^T y + (A')^T z)$

$= \underbrace{(Ax)}_{=b}{}^T y + \underbrace{(A'x)}_{\le b'}{}^T z \le b^T y + (b')^T z = \langle b,y\rangle + \langle b',z\rangle$

$z \ge 0$ [y unrestricted]

prop [duality for max flow]: $G=(V,E)$ capacitated graph [directed], $s \ne t \in V$ capacities $(c_e)_{e\in E} \ge 0$

max flow $\overset{[\text{last lecture}]}{=}$

$$\max \sum_{e:s\to\cdot} f_e - \sum_{e:\cdot\to s} f_e$$
$$\text{st } \sum_{e:v\to\cdot} f_e - \sum_{e:\cdot\to v} f_e = 0, \forall v \ne s,t$$
$$f_e \le c_e$$
$$f_e \ge 0$$

has dual

$$\min \sum_e c_e x_e$$
$$\text{st } d_v \le d_u + x_e \quad e: u\to v \in E$$
$$d_s = 0$$
$$d_t = 1$$
$$x_e \ge 0 \quad \forall e$$

prop [last lecture]: LP captures min cut

150

pf: [rewrite max flow]

$$\max \; -F_t$$
$$\text{st } d_s \,|\, \sum_{e:s\to\cdot} f_e - \sum_{e:\cdot\to s} f_e - F_s = 0$$
$$d_t \,|\, \sum_{e:t\to\cdot} f_e - \sum_{e:\cdot\to t} f_e - F_t = 0$$
$$d_v \,|\, \sum_{e:v\to\cdot} f_e - \sum_{e:\cdot\to v} f_e = 0 \quad \forall v \ne s,t$$
$$x_e \,|\, f_e \le c_e$$
$$f_e \ge 0$$

unconstrained $\longrightarrow d_s$

$x_e \ge 0 \longrightarrow$

do slower

[dual]

$$\min \; d_s \cdot 0 + d_t \cdot 0 + \sum_{v\ne s,t} d_v \cdot 0 + \sum_e c_e x_e$$

st $d_s = 0$ $\longleftarrow$ F_s unconstrained [vs obj]

$d_t = 1$ $\longleftarrow$ F_t constrained [vs obj]

$d_u^{[out]} - d_v^{[in]} + x_e \ge 0$ $\longleftarrow$ $f_e \ge 0$ [vs obj]

$\equiv d_v \le d_u + x_e$ $\quad \overset{f_e}{\underset{u\to v}{\shortparallel}}$

cor: strong duality $\Rightarrow$ max flow$^{[\text{primal}]}$ = min cut$^{[\text{dual}]}$ [max flow always - feasible - bounded]

rmk: - max flow = min cut crucial to flow also $\Rightarrow$ strong duality crucial to LP also

- FF required $\boxed{\text{integral}}$ capacities to prove), strong duality provides proof for $\boxed{\text{arbitrary}}$ capacities

today: linear programming - duality
- example - by hand
  - mechanized
- def
- weak duality - canonical
  - non canonical
- dual of max flow = min cut

next lecture: LP

lost end - page 3 dual [F]
- page 6 dual [F]