# CS4973 Algorithms: Lecture 11 (2022-02-22)

logistics:
- pset 4 due F 17
- exam 1 - 2022-02-28 19:00
  - see piazza
  - thursday exam review

last lecture:
- flows
  - bipartite matching
  - reductions ↗ to max flow

today: flows

thm: bipartite perfect matching in $O(nm)$ time

pf idea: $G=(L \cup R, E) \mapsto G'=(V', E')$

Q: perfect matching in $G$? ↳ require $|L|=|R|$

Q: max flow in $G'$ of value $|L|$?

correctness: clear

complexity: $O(n+m) + O(nm) + O(m)$

pf sketch:

"forward" reduction: creating new problem

$G$: $n$ vertices, $m$ edges $\mapsto G'$: $n'$ vertices $=O(n)$, $m'$ edges $=O(n+m)$

in time $T(n,m) = O(n+m)$

$G$ has perfect matching $\Rightarrow G'$ has flow of value $|L|$

solving new problem:

$T'(n',m')$ to solve new problem

└ $O(n'm') = O(nm)$

re-express in original parameters

"backwards" reduction:

$G$ has perfect matching $\Leftarrow G'$ having $\boxed{\text{integral}}$ flow value $|L|$

$T''(n,m,n',m') = O(m)$

$\square$

Q: can we solve more $\boxed{\text{general}}$ problems?

def: capacitated graph w/ $\boxed{\text{demands}}$:

- capacitated graph $G=(V,E)$
- capacities $(c_e)_{e\in E}$
- demands $d=(d_v)_{v\in V}$ over $\mathbb{Z}$

a circulation is a flow $f=(f_e)_e$ s.t.

capacity $0\le f_e\le c_e \quad \forall e\in E$

conservation: $f^{in}(v)-f^{out}(v)=d_v \quad \forall v\in V$

The circulation problem is to decide if $\boxed{\text{exists}}$ feasible circulation.

rmk: - no distinguished source/sink

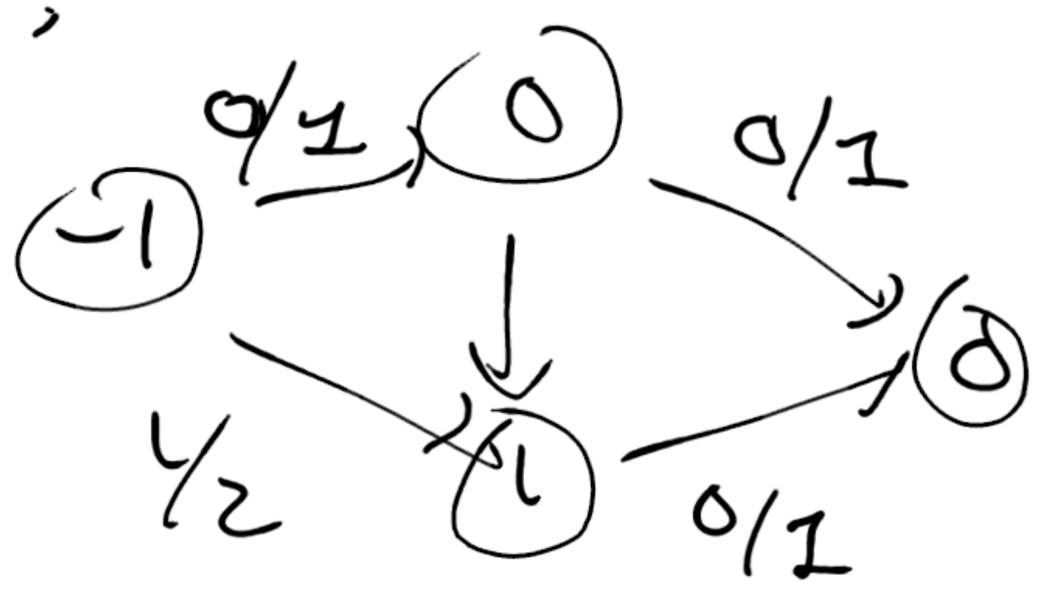
- conservation constraints

$f^{in}(v)-f^{out}(v)=d_v>0$ is $\boxed{\text{demand}}$

$=0$ is $\boxed{\text{conservation}}$

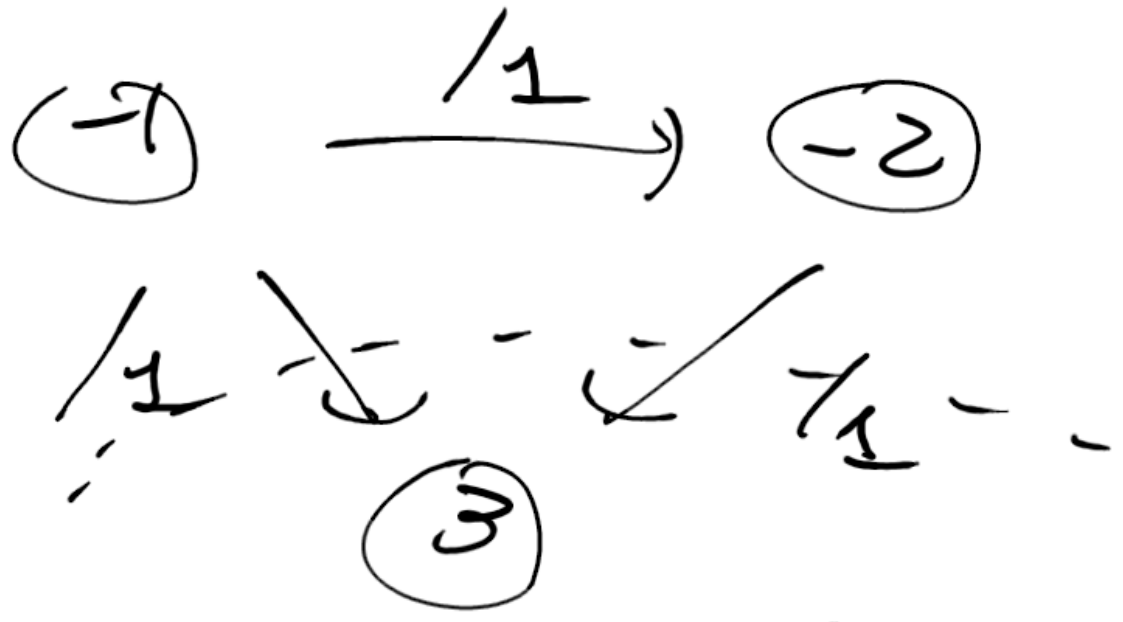
$<0$ is $\boxed{\text{supply}}$

eg:



feasible

eg:

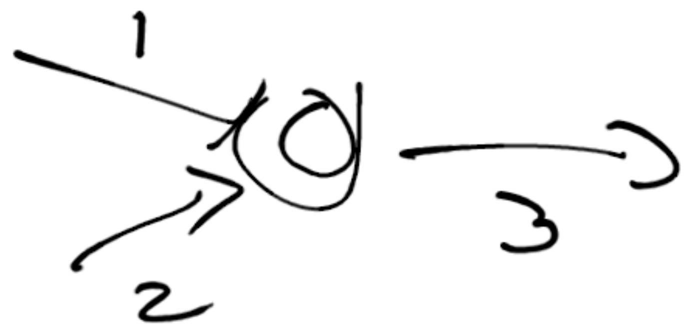


cut value = 2 < demand 3

infeasible

$f^{out}=4$

$f^{in}=7$



lem: feasible circulation exists $\Rightarrow$ $\sum_v d_v = 0$

pf: $0 = f^{in}(V) - f^{out}(V)$

$= -f(V)$

$= -\sum_v \big[\underbrace{f^{in}(v) - f^{out}(v)}_{= d_v}\big]$ □

thm: circulation feasibility in $O(nm)$ time

if all demands, capacities integral, $\boxed{\text{and}}$ circulation feasible, $\Rightarrow$ exists $\boxed{\text{integral}}$ feasible circulation

pf: idea: reduce to max flow

$G$ capacitated graph w/ demands $d$

construction: capacitated graph $G' = (V', E')$

$V' = \{s'\} \cup V \cup \{t'\}$

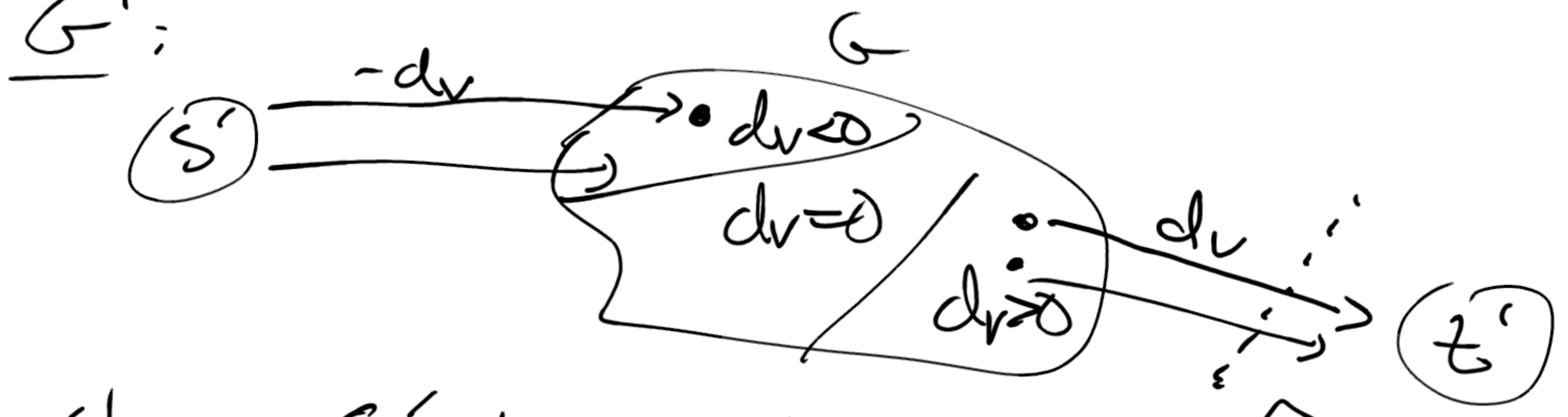
$E' = E$

$\cup \{(s', v) : v \in V \quad d_v < 0\}$

$\cup \{(v, t') : v \in V, \quad d_v > 0\}$

$$c'_{e'} = \begin{cases} c_{e'} & e' \in E \\ -d_v & e' = (s', v) \quad d_v < 0 \\ d_v & e' = (v, t') \quad d_v > 0 \end{cases}$$

$G'$:




clm: $G'$ has $O(n)$ vertices

$O(n+m)$ edges

$O(n+m)$ time to construct

clm: $G$ has feasible circulation $\Rightarrow$

$G'$ has max flow value $D_u$

$\sum_{v : d_v > 0} d_v$

pf: clm: max flow of $G' \le D$

pf: via cut

circulation $f$ in $G$

create $f'$ in $G'$

$$(f')_{e'} = \begin{cases} f_{e'} & e' \in E \\ -d_v & e' = (s', v) \quad d_v < 0 \\ d_v & e' = (v, t') \quad d_v > 0 \end{cases}$$

Clm: $f'$ valid flow

pf: capacity:

$$u \xrightarrow{f_e/c_e \,=\, f'_e/c'_e} v$$

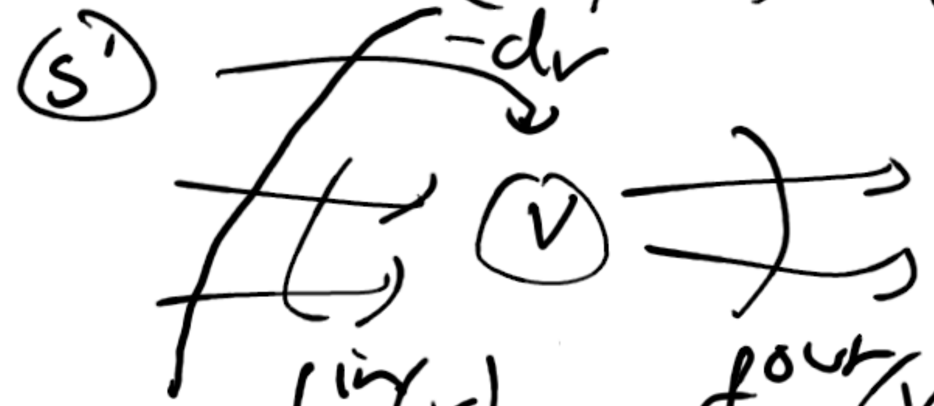
$$s' \xrightarrow{-d_v/-d_v} v$$

$$v \xrightarrow{d_v/d_v} t'$$

conservation

$$f^{in}(v) - f^{out}(v) = d_v$$

$G'$: $d_v < 0$.



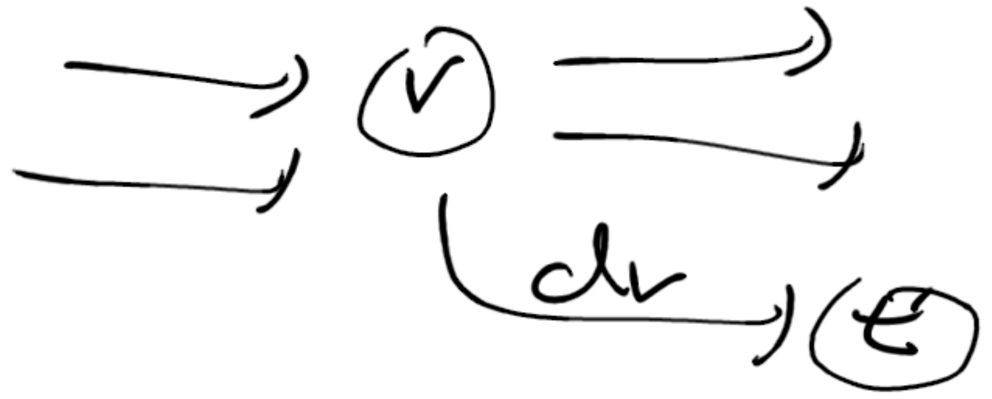

$f^{in}(v)$ $\qquad$ $f^{out}(v)$

$\qquad\qquad\qquad\quad$ $\shortparallel$

$(f')^{in}(v)$ $\qquad$ $(f')^{out}(v)$

$$(f')^{in} = \underset{\substack{\shortparallel \\ d_v + f^{out}}}{f^{in}} + (-d_v) = f^{out} = (f')^{out}$$

$d_v > 0$: analogous




Clm: $f'$ has value $D = \sum_{v: d_v > 0} d_v$

pf:


$$f'(s') = \sum_{v: d_v < 0} -d_v =$$

$$\sum_v d_v = 0$$

clm: $G$ has $\boxed{\text{integral}}$ feasible circulation

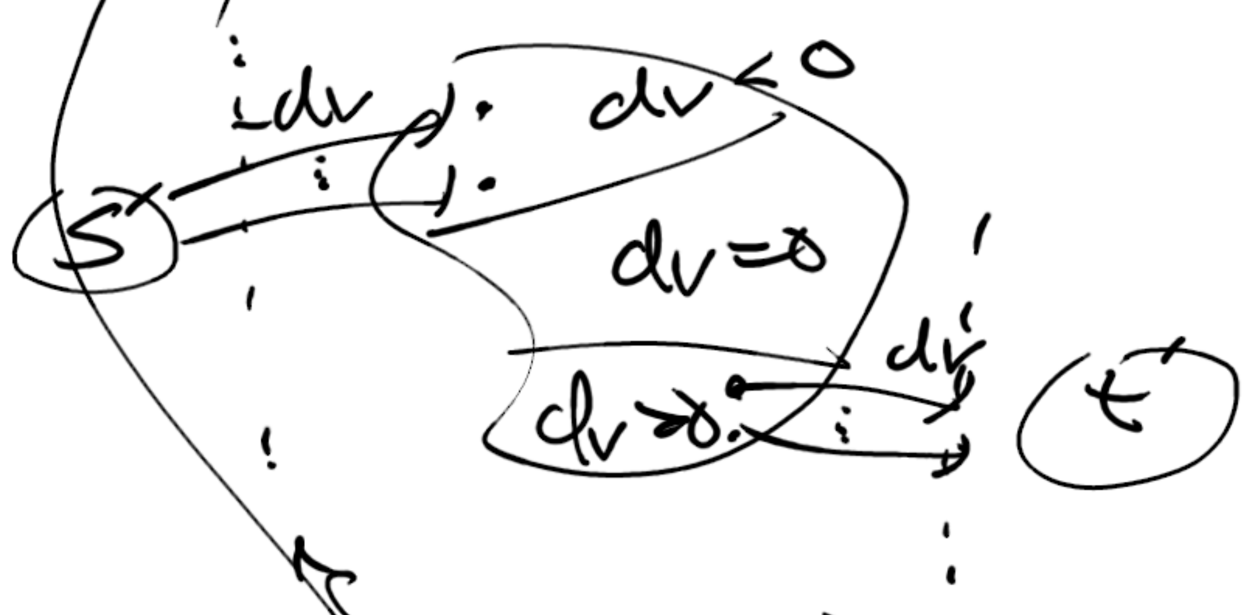
$\Leftarrow$ $G'$ has $\boxed{\text{integral}}$ max flow of value $D = \sum_{v : d_v > 0} d_v$

$\Sigma d_v = 0$

pf: integral flow $f'$ in $G'$

clm: $f'_{e'} = \begin{cases} -d_v & e' = (s', v) \quad d_v < 0 \\ d_v & e' = (v, t') \quad d_v > 0 \end{cases}$

sketch:




cuts are of value $D$

$\Rightarrow$ every edge in cut is saturated □

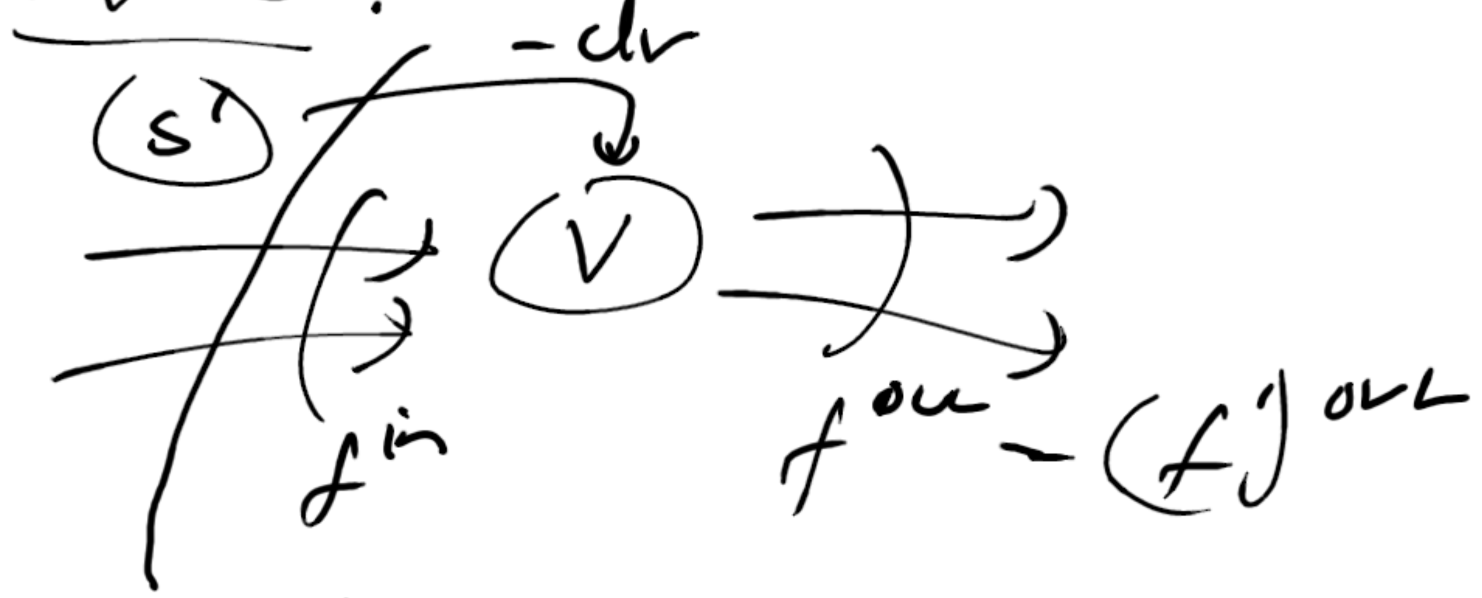
$\Rightarrow$ create circulation $f$ in $G$ by $f_e = f'_e$, $e \in E$

clm: $f$ feasible, integral

pf: capacity = clear

conservation:

$d_v < 0$:




$(f')^{in} = (f')^{out} = f^{out}$

$\|$

$f^{in} + (-d_v) \Rightarrow f^{in} - f^{out} = d_v$

$d_v > 0$: analogous □

algo: construct $G'$ from $G$ $O(m)$

solve $\boxed{\text{integral}}$ max flow in $G'$ $O(nm)$

reconstruct circulation in $G$ ''

correctness: clear $O(m)$ $O(nm)$

complexity: $O(nm)$ □

def: capacitated graphs w/ $\boxed{\text{demands}}$ and $\boxed{\text{lower bounds}}$

capacitated graph $G = (V, E)$

capacities $(c_e)_{e \in E}$ $c_e > 0$

demands $(d_v)_{v \in V}$

lower bounds $(l_e)_{e \in E}$

$0 \le l_e \le c_e$

capacity constraint $0 \le l_e \le f_e \le c_e$

conservation : - - -

the circulation feasibility is - - -

thm: circulation feasibility w/ $\boxed{\text{lower bounds}}$ solvable in $O(\ldots)$ time

↳ integrality - - -

sketch: reduce to circulation w/ $\boxed{\text{no}}$ lower bounds

clm: $f$ feasible in $G$

iff $f = f^l + \tilde{f}$

w/ $f^l$ : $(f^l)_e = l_e$

$\tilde{f}$ : $\tilde{f}$ circulation in $\tilde{G}$ ↖ $\boxed{\text{no}}$ lower bounds

capacity : $0 \le \tilde{f}_e \le c_e - \underbrace{l_e}_{(f^l)_e}$

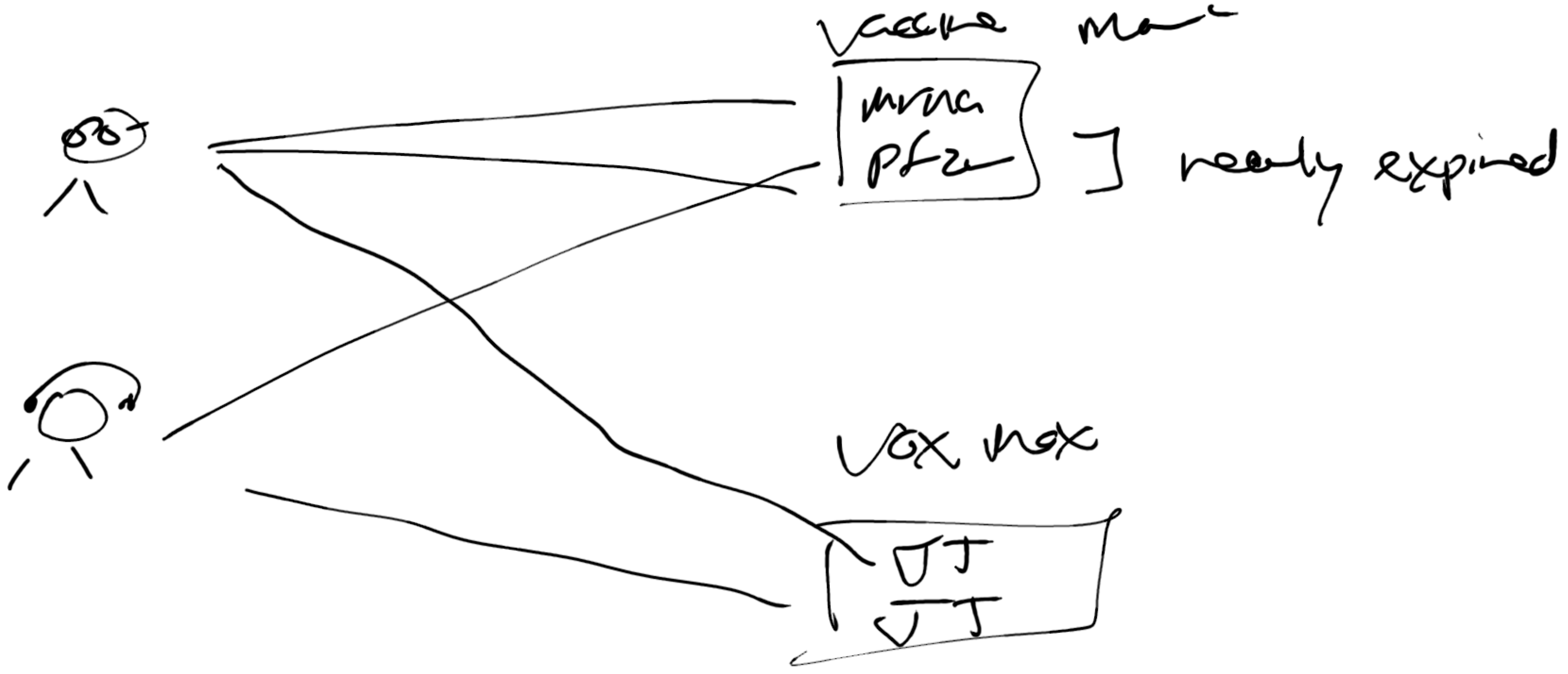
conservation. $f^{in}(v) - f^{out}(v) = d_v$

$= \underbrace{((f^l)^{in} - (f^l)^{out})}_{=: l_v} + \underbrace{((\tilde{f})^{in} - (\tilde{f})^{out})}_{\tilde{d}_v}$

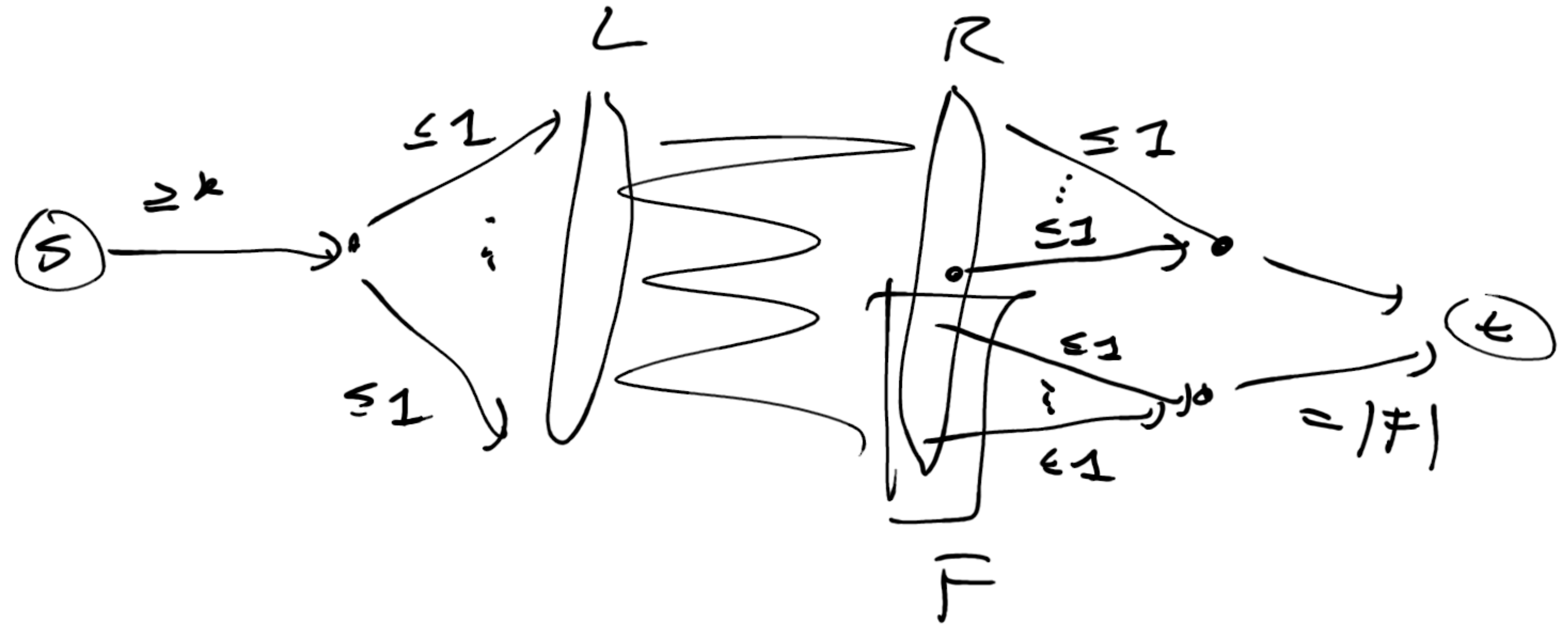
$\Leftarrow$ $\tilde{d}_v = d_v - l_v$

$\square$

Q: can we use all nearly-expired vaccines?




Q: matching size $\geq k$ in bipartite graph $G=(L \cup R, E)$ where $F \subseteq R$ are matched?




today: flow - reductions to maxflow

- circulations w/ demands
- ____________, lower bounds
- bipartite matching, w/ forced matching

next lecture: exam review

logistics: - pset 4 due F/7
- exam 2 02-28 19:00