CS 473: Algorithms, Spring 2021
HW 9 (due Wednesday, April 21th at 8pm)

This homework contains three problems. **Read the instructions for submitting homework on the course webpage**.

Collaboration Policy: For this home work, each student can work in a group with up to three members. Only one solution for each group needs to be submitted. Follow the submission instructions carefully.

For problems that ask for a linear-programming formulation of some problem, a full credit solution requires the following components:

- A list of variables, along with a brief English description of each variable. (Omitting these English descriptions is a Deadly Sin.)
- A linear objective function (expressed either as minimization or maximization, whichever is more convenient), along with a brief English description of its meaning.
- A sequence of linear inequalities (expressed using $\leq$, $=$, or $\geq$, whichever is more appropriate or convenient), along with a brief English description of each constraint.
- A proof that your linear programming formulation is correct, meaning that the optimal solution to the original problem can always be obtained from the optimal solution to the linear program. This may be very short.

It is ***not*** necessary to express the linear program in canonical form, or even in matrix form. Clarity is much more important than formality.

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1. Consider a polyhedron P in n dimensions defined by a set of m inequalities $Ax \leq b$. Let $z \in \mathbb{R}^n$ be a point. Describe a polynomial-time algorithm to check whether z is a basic feasible solution (or equivalently a vertex solution) of P.

2. Consider the following linear program (relaxation) to find a maximum matching in a given bipartite graph $G = (V_1 \cup V_2, E)$. Here variable x_{ij} indicates an edge between nodes $i \in V_1$ and $j \in V_2$.

$$\begin{array}{lll} max: & \sum_{(i,j)\in E} x_{ij} & \\ s.t. & \sum_{j\in V_2:(i,j)\in E} x_{ij} \leq 1, & \forall i \in V_1 \\ & \sum_{i\in V_1:(i,j)\in E} x_{ij} \leq 1, & \forall j \in V_2 \\ & \quad x_{ij} \geq 0 & \forall (i,j) \in E \end{array}$$

   Write the dual of the above linear program. Can you interpret its solution? What do the dual variables represent?

3. You are given the following linear program

$$\begin{array}{ll} max & \boldsymbol{c}^T \boldsymbol{x} \\ s.t. & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array}$$

   and you know that there exists an $\boldsymbol{x}$ that is a feasible solution to the above program. Furthermore, you know that the dual of this program also has a $\boldsymbol{y}$ that is a feasible solution.

   (a) Write the dual program to the above primal program.

   (b) Design a single *linear feasibility formulation* that exactly captures all the *optimal solutions* of both primal and the dual. Show that (x^*, y^*) is feasible in your formulation if and only if x^* is an optimal solution of the primal and y^* is an optimal solution of the dual.