CS 473: Algorithms

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Spring 2021

Inequalities & Randomized QuickSort

Lecture 8

Feb 18, 2021

Most slides are courtesy Prof. Chekuri

Outline

Slick Analysis of Randomized **QuickSort**

Concentration of Mass Around Mean

Markov's Inequality

Chebyshev's Inequality

Chernoff Bound

Randomized **QuickSort**: High Probability Analysis

Part I

Analysis of QuickSort

Recall: Randomized **QuickSort**

Randomized **QuickSort**

1. Pick a pivot element *uniformly at random* from the array.
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
3. Recursively sort the subarrays, and concatenate them.

Theorem

Expected running time of Randomized **QuickSort** *on an array of size* n *is* $O(n \log n)$.

Analysis via Recurrence

1. A: Given array with n *distinct* numbers.
2. $Q(A)$ **:** number of comparisons of randomized **QuickSort** on A. Note that $Q(A)$ is a random variable.
3. X_i: Random variable indicating if picked pivot has rank i in A.

   A^i_{left} and A^i_{right} be the corresponding left and right subarrays.

$$Q(A) = n + \sum_{i=1}^{n} X_i \cdot \left(Q(A^i_{\text{left}}) + Q(A^i_{\text{right}}) \right).$$

Exactly one non-zero X_i. $\mathbf{E}[X_i] = \mathbf{Pr}[\text{pivot has rank } i] = 1/n$.

Independence of Random Variables

Lemma

Random variables X_i *is independent of random variables* $Q(A^i_{left})$ *as well as* $Q(A^i_{right})$, i.e.

$$\mathbf{E}\left[X_i \cdot Q(A^i_{left})\right] = \mathbf{E}[X_i]\,\mathbf{E}\left[Q(A^i_{left})\right]$$
$$\mathbf{E}\left[X_i \cdot Q(A^i_{right})\right] = \mathbf{E}[X_i]\,\mathbf{E}\left[Q(A^i_{right})\right]$$

Proof.

This is because the algorithm, while recursing on $Q(A^i_{\text{left}})$ and $Q(A^i_{\text{right}})$ uses new random coin tosses that are independent of the coin tosses used to decide the first pivot. Only the latter decides value of X_i. □

Analysis via Recurrence

$T(n) = \max_{A:|A|=n} \mathbf{E}[Q(A)]$ be the worst-case expected running time on arrays of size n.

We have, for any A:

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By linearity of expectation, and independence random variables:

$$\begin{aligned} \mathbf{E}\Big[Q(A)\Big] &= n + \textstyle\sum_{i=1}^{n} \mathbf{E}[X_i] \Big(\mathbf{E}\Big[Q(A^i_{\text{left}})\Big] + \mathbf{E}\Big[Q(A^i_{\text{right}})\Big] \Big) \\ &\leq n + \textstyle\sum_{i=1}^{n} \frac{1}{n} \left(T(i-1) + T(n-i) \right) \end{aligned}$$

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Note that above holds for any A of size n. Therefore

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Proof.

(Guess and) Verify by induction. □

Part II

Slick analysis of QuickSort

A Slick Analysis of **QuickSort**

$Q(A)$ **:** number of comparisons done on input array A

1. Rank of an element is its position in the sorted A.
2. R_{ij} **:** event that rank i element is compared with rank j element, for $1 \leq i < j < n$.

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$$Q(A) = \sum_{1 \leq i < j \leq n} X_{ij}$$

and hence by linearity of expectation,

$$\mathbf{E}\Big[Q(A)\Big] = \sum_{1 \leq i < j \leq n} \mathbf{E}\Big[X_{ij}\Big] = \sum_{1 \leq i < j \leq n} \mathbf{Pr}\Big[R_{ij}\Big].$$

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R_{ij} = rank i element is compared with rank j element.

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As such, probability of comparing **5** to **8** is $\mathbf{Pr}[R_{4,7}]$.

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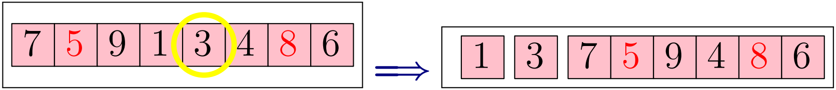

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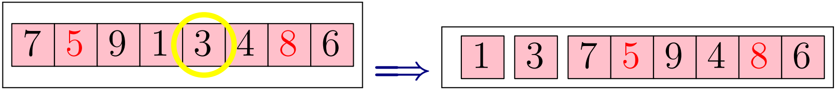

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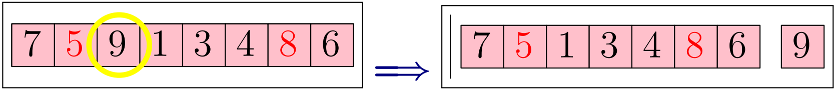

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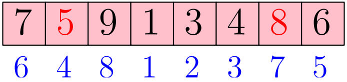

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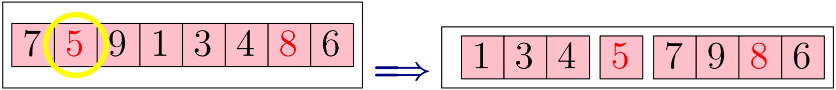

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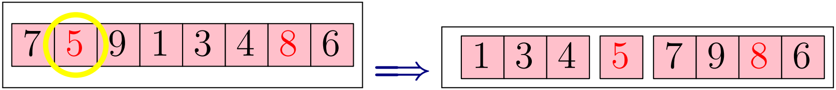
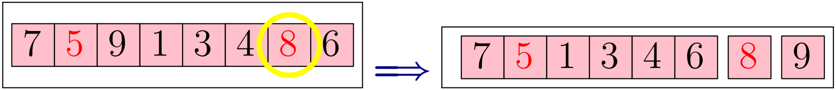


2. If pivot is **8** (rank 7). Bingo!



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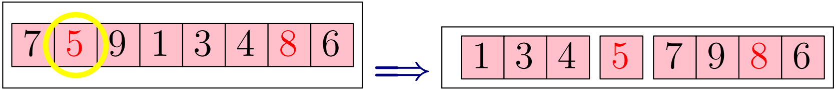

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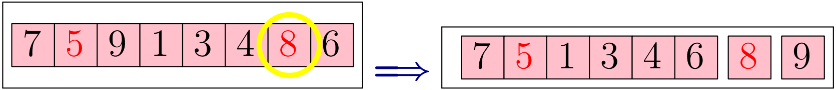
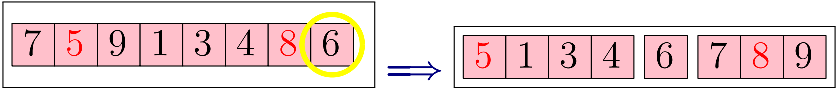


3. If pivot in between the two numbers (say **6** [rank 5]):



5 and **8** will never be compared to each other.

A Slick Analysis of **QuickSort**

Question: What is $\mathbf{Pr[R_{i,j}]}$?

Conclusion:

$R_{i,j}$ happens if and only if:

> ith or jth ranked element is the first pivot out of ith to jth ranked elements.

$$\mathbf{Pr}[R_{i,j}] = Pr[i\text{th or } j\text{th ranked element is the pivot} \mid \text{pivot has rank in } \{i, i+1 \ldots, j-1, j\}]$$

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$$\mathbf{Pr}\left[R_{i,j}\right] = \frac{2}{k} = \frac{2}{j - i + 1}.$$

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Question: What is $\mathbf{Pr}[R_{ij}]$?

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Let $a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n$ be elements of A in sorted order.
Let $S = \{a_i, a_{i+1}, \ldots, a_j\}$

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Observation: a_i is compared with a_j if and only if either a_i or a_j is chosen as a pivot from S at separation... □

A Slick Analysis of **QuickSort**

Continued...

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Observation: a_i is compared with a_j if and only if either a_i or a_j is chosen as a pivot from S at separation.
Observation: Given that pivot is chosen from S the probability that it is a_i or a_j is exactly $2/|S| = 2/(j-i+1)$ since the pivot is chosen uniformly at random from the array. □

A Slick Analysis of **QuickSort**

Continued...

$$\mathbf{E}\Big[Q(A)\Big] = \sum_{1 \le i < j \le n} \mathbf{E}[X_{ij}] = \sum_{1 \le i < j \le n} \mathbf{Pr}[R_{ij}].$$

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$$= 2\sum_{i=1}^{n-1}(H_{n-i+1}-1) \;\le\; 2\sum_{1\le i<n} H_n$$

$$H_k = \sum_{i=1}^{k}\frac{1}{i} = \Theta(\log k)$$

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$$\leq 2nH_n = O(n\log n)$$

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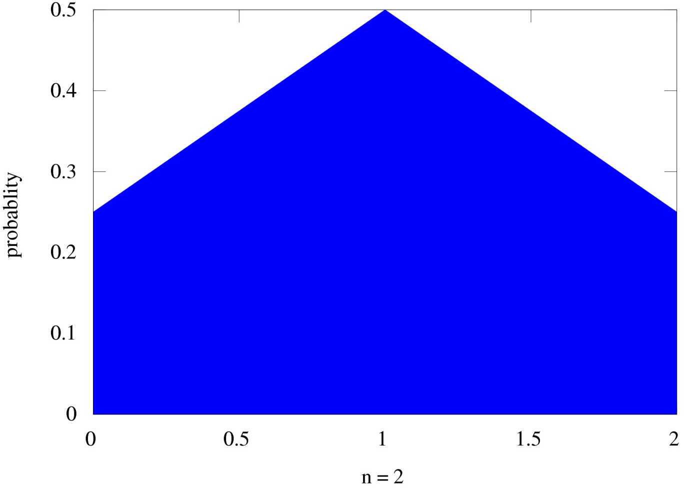
Part III

Inequalities

Massive randomness.. Is not that random.

Consider flipping a fair coin n times independently, head gives $\mathbf{1}$, tail gives zero. How many $\mathbf{1}$s? Binomial distribution: k w.p. $\binom{n}{k} 1/2^n$.



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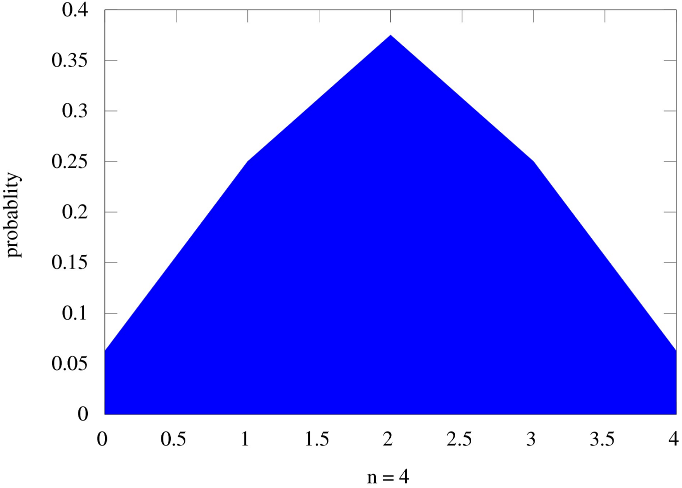

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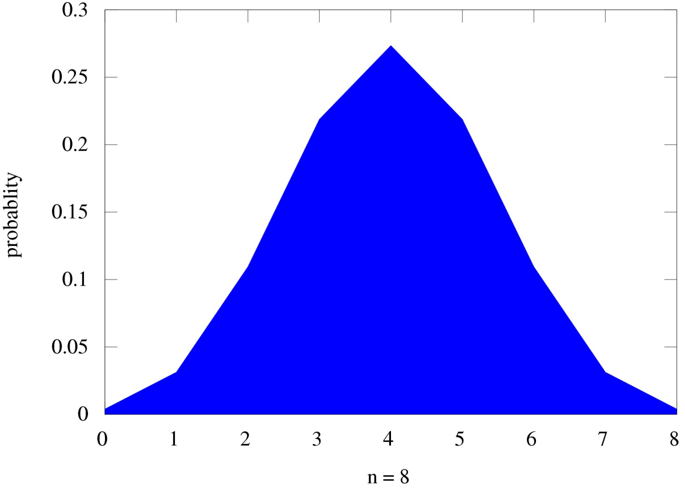

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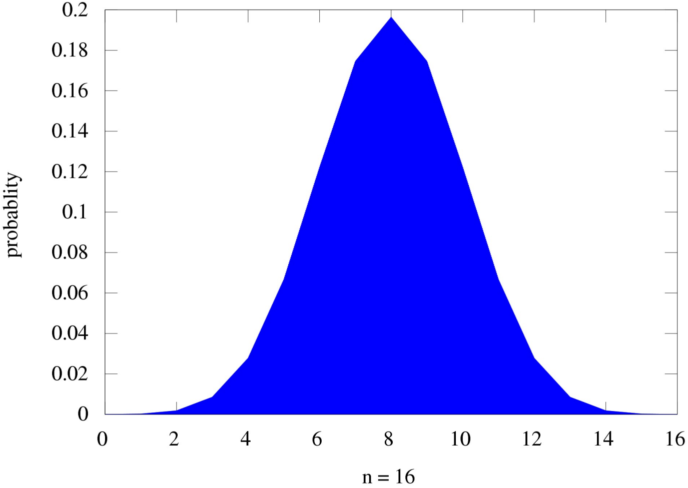

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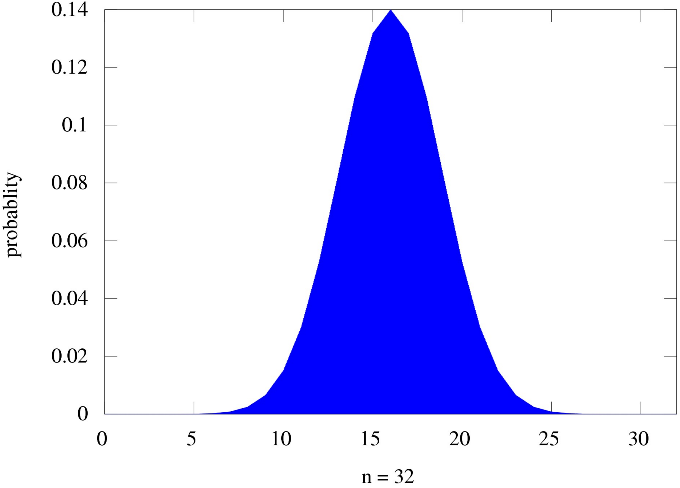

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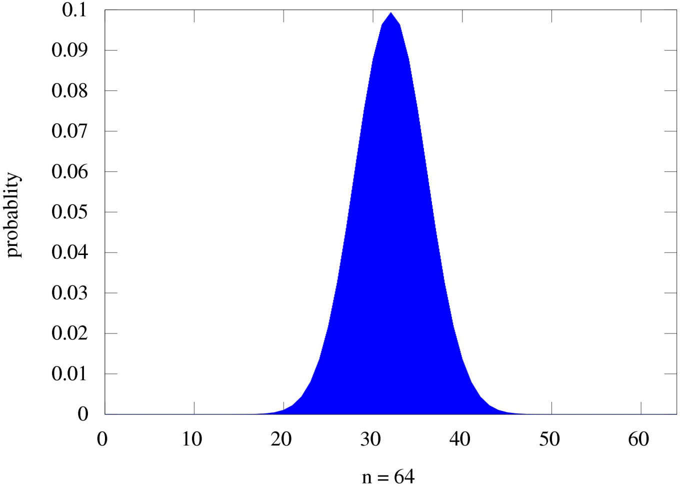

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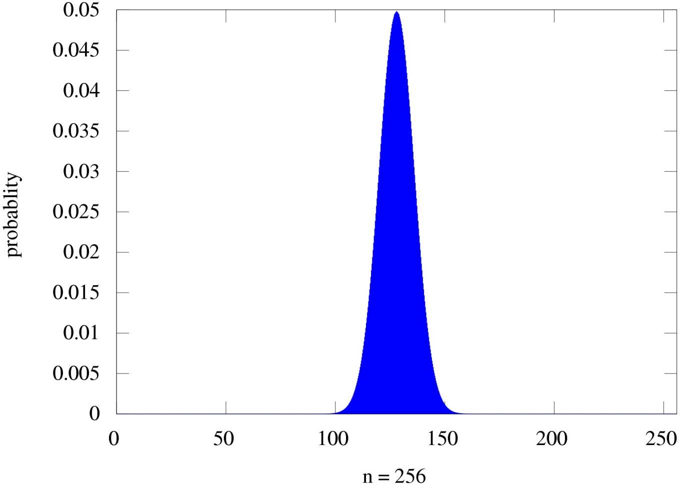

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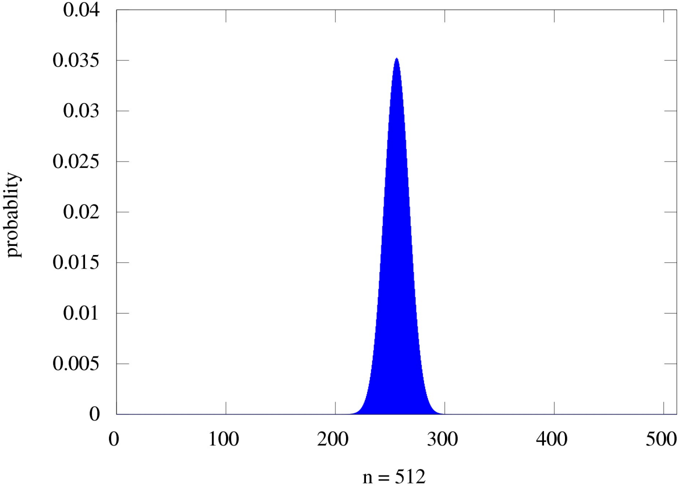

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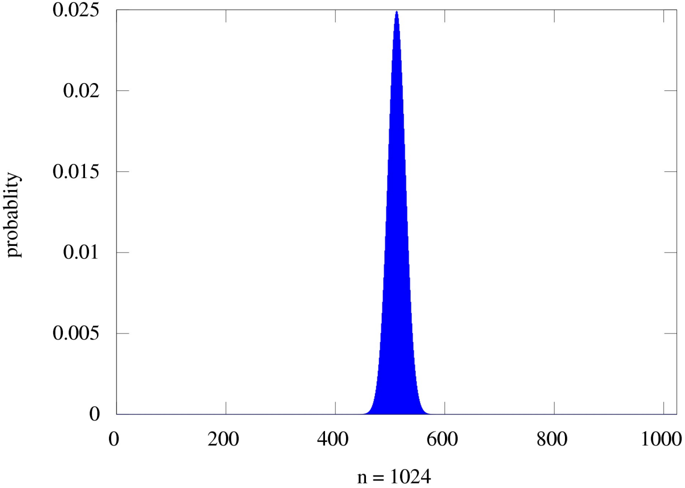

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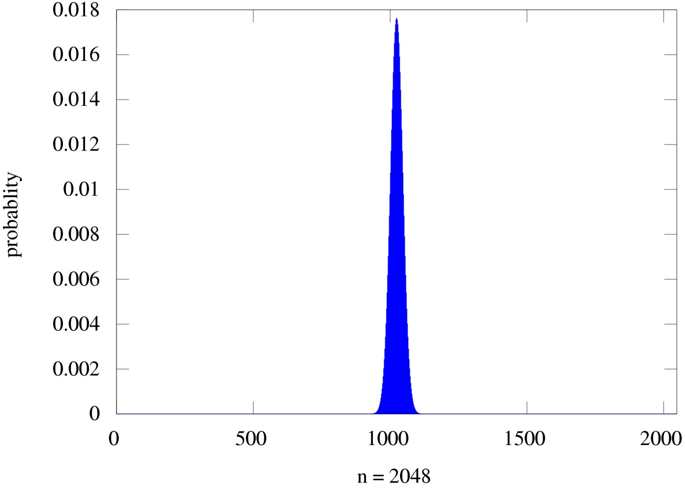

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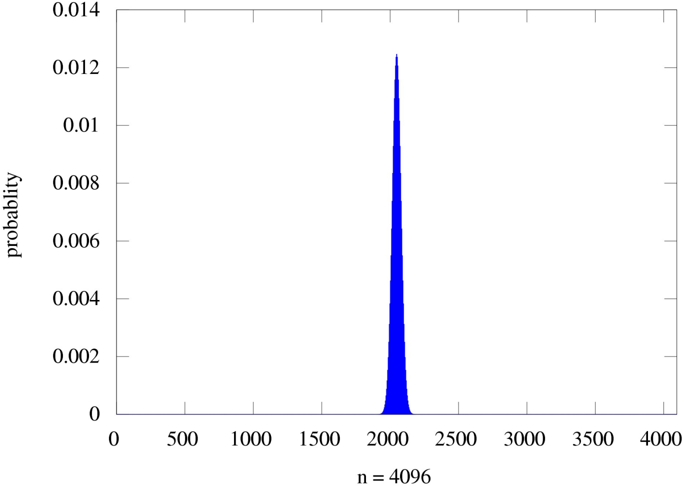

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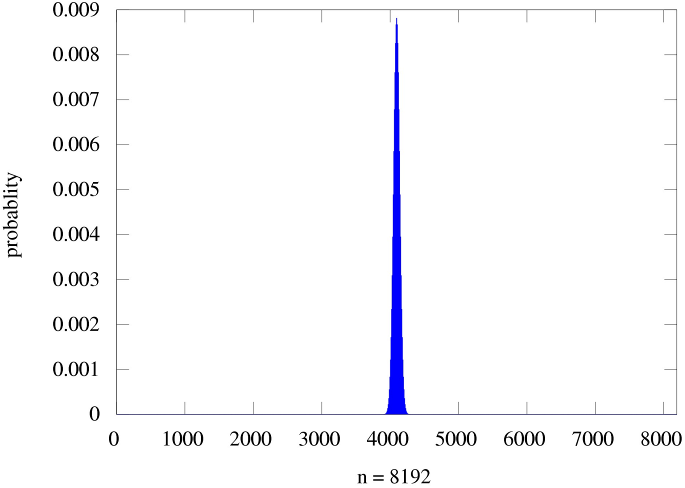
Consider flipping a fair coin n times independently, head gives 1, tail gives zero. How many 1s? Binomial distribution: k w.p. $\binom{n}{k} 1/2^n$.



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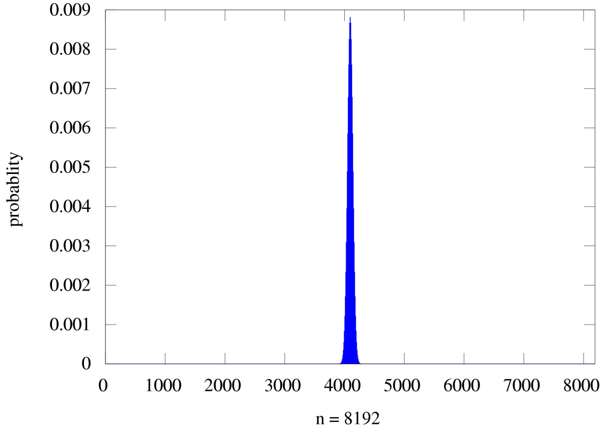


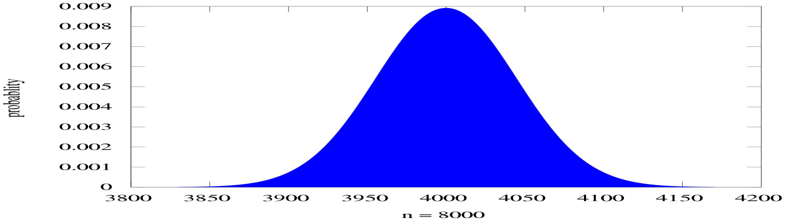
This is known as **concentration of mass**.
This is a very special case of the **law of large numbers**.

Side note...

Law of large numbers (weakest form)...

Informal statement of law of large numbers

For n large enough, the middle portion of the binomial distribution looks like (converges to) the normal/Gaussian distribution.



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Intuitive conclusion

Randomized algorithm are unpredictable in the tactical level, but very predictable in the strategic level.

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Use of well known inequalities in analysis.

Randomized **QuickSort**: A possible analysis

Analysis

- Random variable $Q = \#comparisons$ made by randomized **QuickSort** on an array of n elements.

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Question:

How to find c, or in other words bound $\Pr[Q \geq 10n \log n]$?

Markov's Inequality

Markov's inequality

Let X be a **non-negative** random variable over a probability space $(\Omega, \Pr)$. For any $a > 0$,

$$\Pr[X \geq a] \leq \frac{\mathbf{E}[X]}{a}$$

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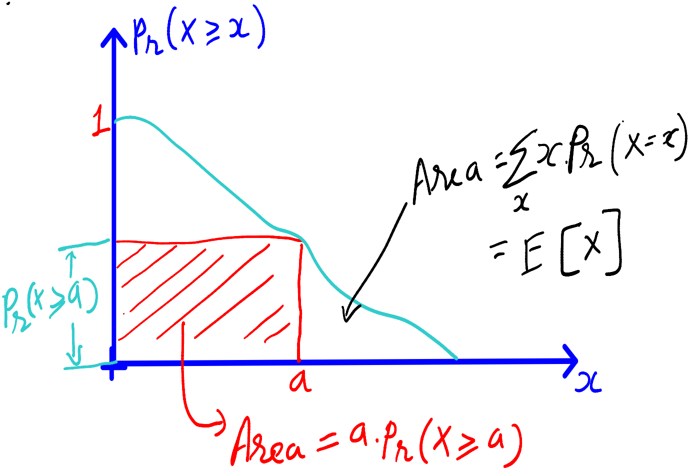
$$\Pr[X \geq a] \leq \frac{\mathbf{E}[X]}{a}$$

Proof:

$$\begin{aligned}
\mathbf{E}[X] &= \sum_{\omega \in \Omega} X(\omega) \Pr[\omega] \\
&\geq \sum_{\omega \in \Omega,\ X(\omega) \geq a} X(\omega) \Pr[\omega] \\
&\geq a \sum_{\omega \in \Omega,\ X(\omega) \geq a} \Pr[\omega] \\
&= a \Pr[X \geq a]
\end{aligned}$$

Markov's Inequality: Proof by Picture



Example: Balls in a bin

- n black and white balls in a bin.
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Question

How large k needs to be before our estimated value p is close to p^*?

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A rough estimate through Markov's inequality.

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For any $k \geq 1$ *and* $p = B/k$, $\Pr[p \geq 2p^*] \leq \frac{1}{2}$

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- $B = \sum_{i=1}^{k} X_i$, then $\mathbf{E}[B] = \sum_{i=1}^{k} \mathbf{E}[X_i] = kp^*$.
- Markov's inequality gives, $\Pr[p \geq 2p^*] =$

$$\Pr\left[\frac{B}{k} \geq 2p^*\right] = \Pr[B \geq 2kp^*] = \Pr[B \geq 2\,\mathbf{E}[B]] \leq \frac{1}{2}$$

Chebyshev's Inequality: Variance

Variance

Given a random variable X over probability space $(\Omega, \Pr)$, variance of X is the measure of how much does it deviate from its mean value. Formally, $Var(X) = \mathbf{E}\left[(X - \mathbf{E}[X])^2\right] = \mathbf{E}\left[X^2\right] - \mathbf{E}[X]^2$

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Chebyshev's Inequality: Variance

Independence

Random variables X and Y are called mutually independent if

$$\forall x, y \in \mathbb{R}, \ \mathbf{Pr}[X = x \wedge Y = y] = \mathbf{Pr}[X = x] \, \mathbf{Pr}[Y = y]$$

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If X and Y are independent random variables then $Var(X + Y) = Var(X) + Var(Y)$.

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If X and Y are mutually independent, then $\mathbf{E}[XY] = \mathbf{E}[X]\,\mathbf{E}[Y]$.

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$Y = (X - E[X])^2$ is a non-negative random variable. Apply Markov's Inequality to Y for a^2.

$$\Pr\left[Y \geq a^2\right] \leq E[Y]/a^2 \quad \Leftrightarrow \quad \Pr\left[(X - E[X])^2 \geq a^2\right] \leq Var(X)/a^2$$
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$\Pr[X \leq \mathbf{E}[X] - a] \leq Var(X)/a^2$ AND
$\Pr[X \geq \mathbf{E}[X] + a] \leq Var(X)/a^2$

Example:Balls in a bin (contd)

Lemma

For $\mathbf{0} < \epsilon < \mathbf{1}$, $k \geq \mathbf{1}$ *and* $p = {}^{B}/_{k}$, $\mathbf{Pr}[|p - p^*| \geq \epsilon] \leq {}^{1}/_{k\epsilon^2}$.

Proof.

- Recall: X_i is $\mathbf{1}$ if i^{th} ball is black, else $\mathbf{0}$, $B = \sum_{i=1}^{k} X_i$. $\mathbf{E}[X_i] = p^*$, $\mathbf{E}[B] = kp^*$. $p = {}^{B}/_{k}$.

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$$\begin{aligned}\Pr[|p - p^*| \geq \epsilon] &= \Pr[|B/k - p^*| \geq \epsilon] \\ &= \Pr[|B - kp^*| \geq k\epsilon] \\ \text{(Chebyshev)} \quad &\leq Var(B)/k^2\epsilon^2 = kp^*(1-p^*)/k^2\epsilon^2 \\ &< 1/k\epsilon^2\end{aligned}$$

□

Chernoff Bound

Lemma

Let $X_1, \ldots, X_k$ be k independent binary random variables such that, for each $i \in [1, k]$, X_i equals 1 w.p. p_i, and 0 w.p. $(1 - p_i)$.

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For any $0 < \delta < 1$, it holds that:

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Proof.

In notes! □

Example:Balls in a bin (Contd.)

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For any $0 < \epsilon < 1$, *and* $k \geq 1$, $\Pr[|p - p^*| \geq \epsilon] \leq 2e^{-\frac{k\epsilon^2}{3}}$.

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Recall: X_i is 1 is i^{th} ball is black, else 0.
$B = \sum_{i=1}^{k} X_i$. $\mathbf{E}[X_i] = p^*$, $\mathbf{E}[B] = kp^*$. $p = {}^{B}/_{k}$.

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&= \mathbf{Pr}\left[|B - kp^*| \geq \left(\frac{\epsilon}{p^*}\right)kp^*\right] \\
(\textit{Chernoff}) &\leq 2e^{-\frac{\epsilon^2}{3{p^*}^2}kp^*} = 2e^{-\frac{k\epsilon^2}{3p^*}} \\
(p^* \leq 1) &\leq 2e^{-\frac{k\epsilon^2}{3}}
\end{aligned}$$

Example Summary

The problem was to estimate the fraction of black balls p^* in a bin filled with white and black balls. Our estimate was $p = \frac{B}{k}$ instead, where out of k draws (with replacement) B balls turns out black.

Markov's Inequality

For any $k \geq 1$, $\Pr[p \geq 2p^*] \leq \frac{1}{2}$

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For any $0 < \epsilon < 1$, and $k \geq 1$, $\Pr[|p - p^*| \geq \epsilon] \leq 1/k\epsilon^2$.

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Part IV

Randomized **QuickSort** (Contd.)

Randomized **QuickSort**: Recall

Input: Array A of n numbers. **Output:** Numbers in sorted order.

Randomized **QuickSort**

1. Pick a pivot element *uniformly at random* from A.
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
3. Recursively sort the subarrays, and concatenate them.

Note: On *every* input randomized **QuickSort** takes $O(n \log n)$ time in expectation. On *every* input it may take $\Omega(n^2)$ time with some small probability.

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Question: With what probability it takes $O(n \log n)$ time?

Randomized **QuickSort**: High Probability Analysis

Informal Statement

Random variable $Q(A) = \#$ comparisons done by the algorithm.

We will show that $\Pr[Q(A) \leq 32 n \ln n] \geq 1 - 1/n^3$.

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If $n = 100$ then this gives $\Pr[Q(A) \leq 32n \ln n] \geq 0.99999$.

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    1. Focus on a single element. Prove that it "participates" in $> 32 \ln n$ levels with probability at most $1/n^4$.
    2. By union bound, any of the n elements participates in $> 32 \ln n$ levels with probability at most

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  2. By union bound, any of the n elements participates in $> 32 \ln n$ levels with probability at most $1/n^3$.
  3. Therefore, all elements participate in $\leq 32 \ln n$ w.p. $(1 - 1/n^3)$.

Randomized **QuickSort**: High Probability Analysis

- If k levels of recursion then kn comparisons.

Randomized **QuickSort**: High Probability Analysis

- If k levels of recursion then kn comparisons.
- Fix an element $s \in A$. We will track it at each level.
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- If $\rho =$#lucky rounds in first k rounds, then $|S_k| \leq (3/4)^\rho n$.

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- If $\rho =$#lucky rounds in first k rounds, then $|S_k| \leq (3/4)^\rho n$.
- For $|S_k| = 1$, $\rho = \log_{4/3} n \leq 4 \ln n$ suffices.

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$$\begin{aligned} \Pr[\rho \leq 4 \ln n] &= \Pr[\rho \leq k/8] \\ &= \Pr[\rho \leq (1-\delta)\mu] \\ (\textit{Chernoff}) \quad &\leq 2e^{\frac{-\delta^2 \mu}{2}} = 2e^{-\frac{9k}{64}} \\ &= 2e^{-4.5 \ln n} \leq \frac{1}{n^4} \end{aligned}$$

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With high probability (i.e., $1 - \frac{1}{n^3}$*) the depth of the recursion of* **QuickSort** *is* $\leq 32 \ln n$*. Due to* n *comparisons in each level, with high probability, the running time of* **QuickSort** *is* $O(n \ln n)$*.*

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Q: How to increase the probability?