Dynamic Programming on Trees

Lecture 4
Feb 4, 2021

Most slides are courtesy Prof. Chekuri
What is Dynamic Programming?

Every recursion can be memoized. Automatic memoization does not help us understand whether the resulting algorithm is efficient or not.

Dynamic Programming:
A recursion that when memoized leads to an efficient algorithm.

Key Questions:
- Given a recursive algorithm, how do we analyze the complexity when it is memoized?
- How do we recognize whether a problem admits a (recursive) dynamic programming based efficient algorithm?
- How do we further optimize time and space of a dynamic programming based algorithm?
Dynamic Programming Template

1. Come up with a recursive algorithm to solve problem
2. Understand the structure/number of the subproblems generated by recursion
3. Memoize the recursion
   - set up compact notation for subproblems
   - set up a data structure for storing subproblems
Dynamic Programming Template

1. Come up with a recursive algorithm to solve problem
2. Understand the structure/number of the subproblems generated by recursion
3. Memoize the recursion
   - set up compact notation for subproblems
   - set up a data structure for storing subproblems
4. Iterative algorithm
   - Understand dependency graph on subproblems
   - Pick an evaluation order (any topological sort of the dependency DAG)
5. Analyze time and space
6. Optimize
**Fact:** Many graph optimization problems are **NP-Hard**

**Fact:** The same graph optimization problems are in $P$ on trees.

Why?
Fact: Many graph optimization problems are NP-Hard

Fact: The same graph optimization problems are in $P$ on trees.

Why?

A significant reason: DP algorithm based on *decomposability*

Powerful methodology for graph algorithms via a formal notion of decomposability called *treewidth* (beyond the scope of this class)
Maximum Independent Set in a Graph

**Definition**

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is an **independent set** (also called a stable set) if for there are no edges between nodes in $S$. That is, if $u, v \in S$ then $(u, v) \not\in E$.

Some independent sets in graph above: $\{D\}, \{A, C\}, \{B, E, F\}$
Maximum Independent Set Problem

Input  Graph  \( G = (V, E) \)

Goal  Find maximum sized independent set in \( G \)
Maximum Weight Independent Set Problem

**Input** Graph $G = (V, E)$, weights $w(v) \geq 0$ for $v \in V$

**Goal** Find maximum weight independent set in $G$
1. No one knows an *efficient* (polynomial time) algorithm for this problem.

2. Problem is **NP-Hard** and it is *believed* that there is no polynomial time algorithm.

**Brute-force algorithm:**

...
Maximum Weight Independent Set Problem

1. No one knows an *efficient* (polynomial time) algorithm for this problem.
2. Problem is **NP-Hard** and it is *believed* that there is no polynomial time algorithm.

**Brute-force algorithm:**
Try all subsets of vertices.
A Recursive Algorithm

Let \( V = \{v_1, v_2, \ldots, v_n\} \).
For a vertex \( u \) let \( N(u) \) be its neighbors.
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**Observation**

\( v_1 \): vertex in the graph.

One of the following two cases is true

- **Case 1** \( v_1 \) is in some maximum independent set.
- **Case 2** \( v_1 \) is in no maximum independent set.

We can try both cases to “reduce” the size of the problem.
A Recursive Algorithm

Let \( V = \{v_1, v_2, \ldots, v_n\} \).

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**Observation**

\( v_1 \): vertex in the graph.

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- Case 2 \( v_1 \) is in no maximum independent set.

We can try both cases to “reduce” the size of the problem

\[
G_1 = G - v_1 \text{ obtained by removing } v_1 \text{ and incident edges from } G \\
G_2 = G - v_1 - N(v_1) \text{ obtained by removing } N(v_1) \cup v_1 \text{ from } G
\]

\[
MIS(G) = \max\{MIS(G_1), MIS(G_2) + w(v_1)\}
\]
A Recursive Algorithm

RecursiveMIS(G):

if G is empty then Output 0
v ← a vertex of G
a = RecursiveMIS(G − v)
b = w(v) + RecursiveMIS(G − v − N(v))
Output max(a, b)
Example
Recursive Algorithms
..for Maximum Independent Set

Running time:

\[ T(n) = T(n - 1) + T\left(n - 1 - \deg(v)\right) + O(1 + \deg(v)) \]

where \( \deg(v) \) is the degree of \( v \). \( T(0) = T(1) = 1 \) is base case.
Recursive Algorithms

..for Maximum Independent Set

Running time:

\[ T(n) = T(n - 1) + T(n - 1 - \text{deg}(v)) + O(1 + \text{deg}(v)) \]

where \( \text{deg}(v) \) is the degree of \( v \). \( T(0) = T(1) = 1 \) is base case.

Worst case is when \( \text{deg}(v) = 0 \) when the recurrence becomes

\[ T(n) = 2T(n - 1) + O(1) \]

Solution to this is \( T(n) = O(2^n) \).
Memoization

We can memoize the recursive algorithm.

**Question:** Does it lead to an efficient algorithm?
Memoization

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What are the sub-problems?
Memoization

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What are the sub-problems? Ans.: Subgraphs (subsets of nodes).

How many are they if $G$ has $n$ nodes to start with?
Memoization

We can memoize the recursive algorithm.

**Question:** Does it lead to an efficient algorithm?

What are the sub-problems? Ans.: Subgraphs (subsets of nodes).

How many are they if $G$ has $n$ nodes to start with? A.: Exponential.

**Exercise:** Show that even when $G$ is a cycle the number of subproblems is exponential in $n$. 
Part I

Maximum Weighted Independent Set in Trees
Maximum Weight Independent Set in a Tree

*Input* Tree $T = (V, E)$ and weights $w(v) \geq 0$ for each $v \in V$

*Goal* Find maximum weight independent set in $T$

Maximum weight independent set in above tree: ??
A Recursive Algorithm

For an arbitrary graph $G$:

1. Number vertices as $v_1, v_2, \ldots, v_n$

2. Find recursively optimum solutions without $v_n$ (recurse on $G - v_n$) and with $v_n$ (recurse on $G - v_n - N(v_n)$ & include $v_n$).

3. Saw that if graph $G$ is arbitrary there was no good ordering that resulted in a small number of subproblems.

What about a tree?
A Recursive Algorithm

For an arbitrary graph $G$:

1. Number vertices as $v_1, v_2, \ldots, v_n$
2. Find recursively optimum solutions without $v_n$ (recurse on $G - v_n$) and with $v_n$ (recurse on $G - v_n - N(v_n)$ & include $v_n$).
3. Saw that if graph $G$ is arbitrary there was no good ordering that resulted in a small number of subproblems.

What about a tree? Natural candidate for $v_n$ is root $r$ of $T$?
Towards a Recursive Solution

Natural candidate for $v_n$ is root $r$ of $T$? Let $\mathcal{O}$ be an optimum solution to the whole problem.

Case $r \notin \mathcal{O}$:

Then $\mathcal{O}$ contains an optimum solution for each subtree of $T$ hanging at a child of $r$.

Case $r \in \mathcal{O}$:

None of the children of $r$ can be in $\mathcal{O}$. $\mathcal{O} - \{r\}$ contains an optimum solution for each subtree of $T$ hanging at a grandchild of $r$.

Subproblems? Subtrees of $T$ rooted at nodes in $T$.

How many of them? $O(n)$
Towards a Recursive Solution

Natural candidate for $v_n$ is root $r$ of $T$? Let $O$ be an optimum solution to the whole problem.

Case $r \not\in O$ : Then $O$ contains an optimum solution for each subtree of $T$ hanging at a child of $r$.

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Subproblems? Subtrees of $T$ rooted at nodes in $T$. How many of them? $O(n)$
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Case $r \in O$ :
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Towards a Recursive Solution

Natural candidate for \( v_n \) is root \( r \) of \( T \)? Let \( \mathcal{O} \) be an optimum solution to the whole problem.

Case \( r \not\in \mathcal{O} \) : Then \( \mathcal{O} \) contains an optimum solution for each subtree of \( T \) hanging at a child of \( r \).

Case \( r \in \mathcal{O} \) : None of the children of \( r \) can be in \( \mathcal{O} \). \( \mathcal{O} - \{r\} \) contains an optimum solution for each subtree of \( T \) hanging at a grandchild of \( r \).

Subproblems? Subtrees of \( T \) rooted at nodes in \( T \).
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Subproblems? Subtrees of $T$ rooted at nodes in $T$.

How many of them? $O(n)$
Example
A Recursive Solution

$T(u)$: subtree of $T$ hanging at node $u$

$OPT(u)$: max weighted independent set value in $T(u)$

$$OPT(u) =$$
A Recursive Solution

$T(u)$: subtree of $T$ hanging at node $u$

$OPT(u)$: max weighted independent set value in $T(u)$

$$OPT(u) = \max \left\{ \sum_{v \text{ child of } u} OPT(v), \quad w(u) + \sum_{v \text{ grandchild of } u} OPT(v) \right\}$$
To evaluate $OPT(u)$ need to have computed values of all children and grandchildren of $u$. Compute $OPT(u)$ bottom up.
Iterative Algorithm

1. To evaluate $OPT(u)$ need to have computed values of all children and grandchildren of $u$. Compute $OPT(u)$ bottom up.

2. What is an ordering of nodes of a tree $T$ to achieve above?

Ans.: Post-order traversal of a tree.
To evaluate $OPT(u)$ need to have computed values of all children and grandchildren of $u$. Compute $OPT(u)$ bottom up.

What is an ordering of nodes of a tree $T$ to achieve above?

Ans.: Post-order traversal of a tree.
Iterative Algorithm

**MIS-Tree** *(T)*:

Let \( v_1, v_2, \ldots, v_n \) be a post-order traversal of nodes of \( T \)

for \( i = 1 \) to \( n \) do

\[
M[v_i] = \max \left( \sum_{v_j \text{ child of } v_i} M[v_j], \ w(v_i) + \sum_{v_j \text{ grandchild of } v_i} M[v_j] \right)
\]

return \( M[v_n] \) (* Note: \( v_n \) is the root of \( T \) *)
Iterative Algorithm

**MIS-Tree**($T$):

Let $v_1, v_2, \ldots, v_n$ be a post-order traversal of nodes of $T$

for $i = 1$ to $n$ do

\[
M[v_i] = \max \left( \sum_{v_j \text{ child of } v_i} M[v_j], w(v_i) + \sum_{v_j \text{ grandchild of } v_i} M[v_j] \right)
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Space:
Iterative Algorithm

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return $M[v_n]$ (* Note: $v_n$ is the root of $T$ *)

Space: $O(n)$ to store the value at each node of $T$

Running time:
Iterative Algorithm

**MIS-Tree**\((T)\):

Let \(v_1, v_2, \ldots, v_n\) be a post-order traversal of nodes of \(T\)

for \(i = 1\) to \(n\) do

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M[v_i] = \max \left( \sum_{v_j \text{ child of } v_i} M[v_j], \quad w(v_i) + \sum_{v_j \text{ grandchild of } v_i} M[v_j] \right)
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return \(M[v_n]\) (* Note: \(v_n\) is the root of \(T\)*)

Space: \(O(n)\) to store the value at each node of \(T\)

Running time:

1. Naive bound: Each \(M[V_i]\) evaluation may take
Iterative Algorithm

\textbf{MIS-Tree} ($T$):

Let $v_1, v_2, \ldots, v_n$ be a post-order traversal of nodes of $T$

\begin{algorithmic}
  \For{$i = 1$ \text{ to } $n$}
    \State $M[v_i] = \max\left(\sum_{v_j \text{ child of } v_i} M[v_j], w(v_i) + \sum_{v_j \text{ grandchild of } v_i} M[v_j]\right)$
  \EndFor
  \Return $M[v_n]$ (* Note: $v_n$ is the root of $T$ *)
\end{algorithmic}

\textbf{Space:} $O(n)$ to store the value at each node of $T$

\textbf{Running time:}

1. Naive bound: Each $M[V_i]$ evaluation may take $O(n)$. 

MIS-Tree($T$):

Let $v_1, v_2, \ldots, v_n$ be a post-order traversal of nodes of $T$
for $i = 1$ to $n$ do

\[
M[v_i] = \max \left( \sum_{\text{child of } v_i} M[v_j], \ w(v_i) + \sum_{\text{grandchild of } v_i} M[v_j] \right)
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return $M[v_n]$ (* Note: $v_n$ is the root of $T$ *)

Space: $O(n)$ to store the value at each node of $T$

Running time:

1. Naive bound: Each $M[V_i]$ evaluation may take $O(n)$. There are $n$ such evaluations – $O(n^2)$. 
Iterative Algorithm

**MIS-Tree**\((T)\):

Let \(v_1, v_2, \ldots, v_n\) be a post-order traversal of nodes of \(T\)

for \(i = 1\) to \(n\) do

\[
M[v_i] = \max \left( \sum_{v_j \text{ child of } v_i} M[v_j], \right.
\]

\[
\left. w(v_i) + \sum_{v_j \text{ grandchild of } v_i} M[v_j] \right)
\]

return \(M[v_n]\) (* Note: \(v_n\) is the root of \(T\) *)

**Space:** \(O(n)\) to store the value at each node of \(T\)

**Running time:**

1. Naive bound: Each \(M[V_i]\) evaluation may take \(O(n)\). There are \(n\) such evaluations – \(O(n^2)\).

2. Better bound: Value \(M[v_j]\) is accessed by who all?
Iterative Algorithm

**MIS-Tree** \((T)\):

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M[v_i] = \max \left( \sum_{v_j \text{ child of } v_i} M[v_j], w(v_i) + \sum_{v_j \text{ grandchild of } v_i} M[v_j] \right)
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return \(M[v_n]\) (* Note: \(v_n\) is the root of \(T\) *)

**Space:** \(O(n)\) to store the value at each node of \(T\)

**Running time:**

1. Naive bound: Each \(M[V_i]\) evaluation may take \(O(n)\). There are \(n\) such evaluations – \(O(n^2)\).

2. Better bound: Value \(M[v_j]\) is accessed by who all? Parent and grand-parent. So in total
Iterative Algorithm

**MIS-Tree** ($T$):

Let $v_1, v_2, \ldots, v_n$ be a post-order traversal of nodes of $T$

for $i = 1$ to $n$ do

$$M[v_i] = \max \left( \sum_{v_j \text{ child of } v_i} M[v_j], \right.$$ \left. w(v_i) + \sum_{v_j \text{ grandchild of } v_i} M[v_j] \right)$$

return $M[v_n]$ (* Note: $v_n$ is the root of $T$ *)

**Space:** $O(n)$ to store the value at each node of $T$

**Running time:**

1. Naive bound: Each $M[V_i]$ evaluation may take $O(n)$. There are $n$ such evaluations – $O(n^2)$.

Why did DP work on trees?

Each node (including the root) is a separator!

**Definition**

Given a graph $G = (V, E)$ a set of nodes $S \subseteq V$ is a separator for $G$ if $G - S$ has at least two connected components.
Why did DP work on trees?

Each node (including the root) is a separator!

**Definition**

Given a graph $G = (V, E)$ a set of nodes $S \subseteq V$ is a separator for $G$ if $G - S$ has at least two connected components.

**Definition**

$S$ is a balanced separator if each connected component of $G - S$ has at most $2|V(G)|/3$ nodes.
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$S$ is a balanced separator if each connected component of $G - S$ has at most $2|V(G)|/3$ nodes.

**Exercise:** Prove that every tree $T$ has a balanced separator consisting of a single node.
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Definition

$S$ is a balanced separator if each connected component of $G - S$ has at most $2|V(G)|/3$ nodes.

Exercise: Prove that every tree $T$ has a balanced separator consisting of a single node.

Aside: $O(2^{\sqrt{n}})$ algorithm to find MIS in planar graphs using, (i) balanced-separators, (ii) DP algorithm on trees.
Part II

Minimum Dominating Set in Trees
Minimum Dominating Set in a Graph

Definition
Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is a dominating set if for all $v \in V$, either $v \in S$ or a neighbor of $v$ is in $S$.

Some dominating sets in graph above: $\{A, B, C, D, E, F\}$,
Input: Graph $G = (V, E)$, weights $w(v) \geq 0$ for $v \in V$

Goal: Find minimum weight dominating set in $G$
Minimum Weight Dominating Set Problem

Input Graph $G = (V, E)$, weights $w(v) \geq 0$ for $v \in V$

Goal Find minimum weight dominating set in $G$

NP-Hard problem
Minimum Weight Dominating Set in a Tree

**Input** Tree $T = (V, E)$ and weights $w(v) \geq 0$ for each $v \in V$

**Goal** Find minimum weight dominating set in $T$

Minimum weight dominating set in above tree: ??
$r$ is root of $T$. Let $O$ be an optimum solution for $T$.

Case $r \not\in O$: Then $O$ must contain some child of $r$. Which one?
Recursive Algorithm

$r$ is root of $T$. Let $O$ be an optimum solution for $T$.

Case $r \notin O$: Then $O$ must contain some child of $r$. Which one?

Case $r \in O$: None of the children of $r$ need to be in $O$ because $r$ can dominate them. However, they may have to be.
Recursive Algorithm

\( r \) is root of \( T \). Let \( \mathcal{O} \) be an optimum solution for \( T \).

Case \( r \notin \mathcal{O} \): Then \( \mathcal{O} \) must contain some child of \( r \). Which one?

Case \( r \in \mathcal{O} \): None of the children of \( r \) need to be in \( \mathcal{O} \) because \( r \) can dominate them. However, they may have to be.

Issue 1: In both cases it is not feasible to express \(|\mathcal{O}|\) easily as optimum solution values of children or descendants of \( r \).
Recursive Algorithm

$r$ is root of $T$. Let $O$ be an optimum solution for $T$.

Case $r \notin O$: Then $O$ must contain some child of $r$. Which one?

Case $r \in O$: None of the children of $r$ need to be in $O$ because $r$ can dominate them. However, they may have to be.

Issue 1: In both cases it is not feasible to express $|O|$ easily as optimum solution values of children or descendants of $r$.

Issue 2: Removing $r$ decomposes $T$ into subtrees rooted at children of $r$. However, not easy to decompose problem structure recursively. Problems at children of $r$ are dependent. Need to introduce additional variable(s).
Let $u_1, u_2, \ldots, u_k$ be children of root $r$ of $T$.

What “information” do $Tu_1, \ldots, Tu_k$ need to know about $r$’s status in an optimum solution in order to become “independent”?
Recursive Algorithm: Understanding Dependence

Let $u_1, u_2, \ldots, u_k$ be children of root $r$ of $T$

What “information” do $T_{u_1}, \ldots, T_{u_k}$ need to know about $r$’s status in an optimum solution in order to become “independent”

- Whether $r$ is included in the solution
- If $r$ is not included then which of the children is going to cover it. Equivalently, $T_{u_i}$ needs to know whether it should cover $r$ or some other child will.
Recursive Algorithm: Introducing Variables

- **u**: node in tree
- **pi**: boolean variable to indicate whether parent is in solution. \( pi = 0 \) means parent is not included. \( pi = 1 \) means it is included.
- **cp**: boolean variable to indicate whether node needed to cover parent. \( cp = 1 \) means parent needs to be covered. \( cp = 0 \) means not needed.
Recursive Algorithm: Introducing Variables

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  \( pi = 0 \) means parent is not included.  \( pi = 1 \) means it is included.
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  \( cp = 1 \) means parent needs to be covered.  \( cp = 0 \) means not needed.

**OPT**\((u, pi, cp)\): value of minimum dominating set in \( T_u \) with booleans \( pi \) and \( cp \) with meaning above.
Recursive Algorithm: Sub-problem

- $u$: node in tree
- $pi$: indicates if the parent is included or not.
- $cp$: indicates if the parent needs to be covered or not.

$OPT(u, pi, cp)$: opt value in $T_u$ with $pi$ and $cp$ as above.

$OPT(u, 0, 0)$: opt value in $T_u$ when parent of $u$ is not included and $u$ need not cover it.
Recursive Algorithm: Sub-problem

- **u**: node in tree
- **pi**: indicates if the parent is included or not.
- **cp**: indicates if the parent needs to be covered or not.

**OPT(u, pi, cp)**: opt value in $T_u$ with **pi** and **cp** as above.

**OPT(u, 0, 0)**: opt value in $T_u$ when parent of **u** is not included and **u** need not cover it.

**OPT(u, 0, 1)**: opt value in $T_u$ when parent of **u** is not included and **u** need to cover it.
Recursive Algorithm: Sub-problem

- **u**: node in tree
- **pi** indicates if the parent is included or not.
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**OPT(u, pi, cp)**: opt value in $T_u$ with $pi$ and $cp$ as above.

**OPT(u, 0, 0)**: opt value in $T_u$ when parent of $u$ is not included and $u$ need not cover it.

**OPT(u, 0, 1)**: opt value in $T_u$ when parent of $u$ is not included and $u$ need to cover it.

**OPT(u, 1, 0)**: opt value in $T_u$ when parent of $u$ is included and $u$ need not cover it.
Recursive Algorithm: Sub-problem

- **u**: node in tree
- **pi**: indicates if the parent is included or not.
- **cp**: indicates if the parent needs to be covered or not.

**OPT**(u, pi, cp): opt value in **Tu** with pi and cp as above.

**OPT**(u, 0, 0): opt value in **Tu** when parent of u is not included and u need not cover it.
**OPT**(u, 0, 1): opt value in **Tu** when parent of u is not included and u need to cover it.
**OPT**(u, 1, 0): opt value in **Tu** when parent of u is included and u need not cover it.
**OPT**(u, 1, 1): NOT NEEDED!
Recursive Algorithm: Sub-problem

- **u**: node in tree
- **pi**: indicates if the parent is included or not.
- **cp**: indicates if the parent needs to be covered or not.

**OPT(u, pi, cp)**: opt value in $T_u$ with **pi** and **cp** as above.

**OPT(u, 0, 0)**: opt value in $T_u$ when parent of **u** is not included and **u** need not cover it.

**OPT(u, 0, 1)**: opt value in $T_u$ when parent of **u** is not included and **u** need to cover it.

**OPT(u, 1, 0)**: opt value in $T_u$ when parent of **u** is included and **u** need not cover it.

**OPT(u, 1, 1)**: NOT NEEDED!

**OPT(r, 0, 0)**: value of minimum dominating set in $T$. 
Recursive Solution

Can we express $OPT(u, pi, cp)$ recursively via children of $u$?
Recursive Solution

Can we express $OPT(u, pi, cp)$ recursively via children of $u$?

$OPT(u, 0, 0)$: Value of a minimum dominating set in $T_u$ where we assume that $u$'s parent is not included and $u$ does not need to cover its parent.
Recursive Solution

Can we express $OPT(u, pi, cp)$ recursively via children of $u$?

$OPT(u, 0, 0)$: Value of a minimum dominating set in $T_u$ where we assume that $u$'s parent is not included and $u$ does not need to cover its parent. Let $C_u$ be children of $u$.

Case $u$ is included: Then $u$ does not need to be covered by any child.

$$OPT(u, 0, 0) = w(u) + \sum_{v \in C_u} OPT(v, 1, 0)$$

Case $u$ is not included: Then $u$ needs to be covered by some child. We do a min over all children.

$$OPT(u, 0, 0) = \min_{v \in C_u} (OPT(v, 0, 1) + \sum_{v' \in C_u - v} OPT(v', 0, 0))$$

Since one of these cases has to be true, we take the min of the values in the above two cases to compute $OPT(u, 0, 0)$.
Recursive Solution

Can we express $OPT(u, pi, cp)$ recursively via children of $u$?

$OPT(u, 0, 0)$: Value of a minimum dominating set in $T_u$ where we assume that $u$'s parent is not included and $u$ does not need to cover its parent. Let $C_u$ be children of $u$.

Case $u$ is included: Then $u$ does not need to be covered by any child. Include $u$ and recurse.

$$OPT(u, 0, 0) = w(u) + \sum_{v \in C_u}$$
Recursive Solution

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$$OPT(u, 0, 0) = w(u) + \sum_{v \in C_u} OPT(v, 1, 0)$$

Case $u$ is not included: Then $u$ needs to be covered by some child. We do a min over all children.

$$OPT(u, 0, 0) = \min_{v \in C_u}$$
Recursive Solution

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Case $u$ is included: Then $u$ does not need to be covered by any child. Include $u$ and recurse.

$$OPT(u, 0, 0) = w(u) + \sum_{v \in C_u} OPT(v, 1, 0)$$

Case $u$ is not included: Then $u$ needs to be covered by some child. We do a min over all children.

$$OPT(u, 0, 0) = \min_{v \in C_u} (OPT(v, 0, 1) + \sum_{v' \in C_u - v} OPT(v', 0, 0))$$
Recursive Solution

Can we express $OPT(u, pi, cp)$ recursively via children of $u$?

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$$OPT(u, 0, 0) = w(u) + \sum_{v \in C_u} OPT(v, 1, 0)$$

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Since one of these cases has to be true, we take the min of the values in the above two cases to compute $OPT(u, 0, 0)$. 
Recursive Solution

\[ \text{OPT}(u, 0, 1) : \text{Value of a minimum dominating set in } T_u \text{ where we assume that } u \text{'s parent is not included and } u \text{ needs to cover its parent. Let } C_u \text{ be children of } u. \]
Recursive Solution

**OPT(\(u, 0, 1\))**: Value of a minimum dominating set in \(T_u\) where we assume that \(u\)’s parent is not included and \(u\) needs to cover its parent. Let \(C_u\) be children of \(u\).

**Case \(u\) is included**: Then \(u\) does not need to covered by any child. Include \(u\) and recurse.

\[
OPT(u, 0, 1) = w(u) + \sum_{v \in C_u} OPT(v, 1, 0)
\]
Recursive Solution

\( \text{OPT}(u, 0, 1) \) : Value of a minimum dominating set in \( T_u \) where we assume that \( u \)'s parent is not included and \( u \) needs to cover its parent. Let \( C_u \) be children of \( u \).

Case \( u \) is included: Then \( u \) does not need to be covered by any child. Include \( u \) and recurse.

\[ \text{OPT}(u, 0, 1) = w(u) + \sum_{v \in C_u} \text{OPT}(v, 1, 0) \]

Case \( u \) is not included:
Recursive Solution

\( OPT(u, 0, 1) \): Value of a minimum dominating set in \( T_u \) where we assume that \( u \)'s parent is not included and \( u \) needs to cover its parent. Let \( \mathcal{C}_u \) be children of \( u \).

Case \( u \) is included: Then \( u \) does not need to covered by any child. Include \( u \) and recurse.

\[
OPT(u, 0, 1) = w(u) + \sum_{v \in \mathcal{C}_u} OPT(v, 1, 0)
\]

Case \( u \) is not included: This does not arise because \( u \) has to cover its parent.
Recursive Solution

$OPT(u, 1, 0)$ : Value of a minimum dominating set in $T_u$ where we assume that $u$’s parent is included and $u$ does not need to cover its parent. Let $C_u$ be children of $u$. 

Case $u$ is included: Then $u$ does not need to be covered by any child. Include $u$ and recurse.

$OPT(u, 1, 0) = w(u) + \sum_{v \in C_u} OPT(v, 1, 0)$

Case $u$ is not included: $u$’s parent is included. Now, does $u$ need to be covered by its children? No. Thus we have,

$OPT(u, 1, 0) = \sum_{v \in C_u} OPT(v, 0, 0)$

Take the min of the values in the above two cases to compute $OPT(u, 1, 0)$.

Caution: Not including $u$ may appear to be always advantageous but it is not true.
Recursive Solution

\( \text{OPT}(u, 1, 0) \) : Value of a minimum dominating set in \( T_u \) where we assume that \( u \)'s parent is included and \( u \) does not need to cover its parent. Let \( C_u \) be children of \( u \).

Case \( u \) is included: Then \( u \) does not need to be covered by any child. Include \( u \) and recurse.

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Take the min of the values in the above two cases to compute \( \text{OPT}(u, 1, 0) \).

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\textit{OPT}(u, 1, 0) = w(u) + \sum_{v \in C_u} \textit{OPT}(v, 1, 0)
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Case \( u \) is not included: \( u \)'s parent is included. Now, does \( u \) need to be covered by its children? No. Thus we have,

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\]

Take the min of the values in the above two cases to compute \( \textit{OPT}(u, 1, 0) \).
Recursive Solution

**OPT**\((u, 1, 0)\) : Value of a minimum dominating set in \(T_u\) where we assume that \(u\)'s parent is included and \(u\) does not need to cover its parent. Let \(C_u\) be children of \(u\).

**Case** \(u\) is included: Then \(u\) does not need to be covered by any child.

- Include \(u\) and recurse.
- \(\text{OPT}(u, 1, 0) = w(u) + \sum_{v \in C_u} \text{OPT}(v, 1, 0)\)

**Case** \(u\) is not included: \(u\)'s parent is included. Now, does \(u\) need to be covered by its children? No. Thus we have,

\(\text{OPT}(u, 1, 0) = \sum_{v \in C_u} \text{OPT}(v, 0, 0)\)

Take the min of the values in the above two cases to compute \(\text{OPT}(u, 1, 0)\).

**Caution:** Not including \(u\) may appear to be always advantageous but it is not true.
Recursive Solution

\( OPT(u, 1, 1) \): Value of a minimum dominating set in \( T_u \) where we assume that \( u \)'s parent is included and \( u \) needs to cover its parent.

This subproblem does not make sense since if \( u \)'s parent is included then \( u \) does not need to cover it.
Base Cases

Leaves are base cases. If $u$ is a leaf.

\[ \text{OPT}(u, 0, 0) = \]
Base Cases

Leaves are base cases. If \( u \) is a leaf.

- \( OPT(u, 0, 0) = w(u) \)
- \( OPT(u, 0, 1) = \)
Base Cases

Leaves are base cases. If $u$ is a leaf.

- $OPT(u, 0, 0) = w(u)$
- $OPT(u, 0, 1) = w(u)$
- $OPT(u, 1, 0) =$
Base Cases

Leaves are base cases. If $u$ is a leaf.

- $OPT(u, 0, 0) = w(u)$
- $OPT(u, 0, 1) = w(u)$
- $OPT(u, 1, 0) = 0$
DP Algorithm

- Minimum weight dominating set value in $T$ is $OPT(r, 0, 0)$
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To compute $OPT(r, 0, 0)$ we need to compute recursively $OPT(u, 0, 0), OPT(u, 0, 1), OPT(u, 1, 0)$ for all $u \in T$. Thus number of subproblems is $O(n)$. 

Nodes should be traversed in what order? Ans.: bottom up from leaves to root. In particular? Ans.: post-order traversal.
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To compute $OPT(r, 0, 0)$ we need to compute recursively $OPT(u, 0, 0), OPT(u, 0, 1), OPT(u, 1, 0)$ for all $u \in T$. Thus number of subproblems is $O(n)$.

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DP Algorithm

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**DP Algorithm**

- Minimum weight dominating set value in $T$ is $OPT(r, 0, 0)$
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To compute $OPT(r, 0, 0)$ we need to compute recursively $OPT(u, 0, 0)$, $OPT(u, 0, 1)$, $OPT(u, 1, 0)$ for all $u \in T$. Thus number of subproblems is $O(n)$.

Nodes should be traversed in what order? Ans.: bottom up from leaves to root.

In particular? Ans.: post-order traversal.
Iterative Algorithm

**DominatingSet-Tree**($T$):

Let $v_1, v_2, \ldots, v_n$ be a post-order traversal of nodes of $T$

Allocate array $M[1..n, 0..1, 0..1]$ to store $OPT(v_i, pi, cp)$ values

for $i = 1$ to $n$ do

Compute $OPT(v_i, 0, 0)$, $OPT(v_i, 1, 0)$ and $OPT(v_i, 0, 1)$ using values of children of $v_i$ stored in $M$,
or via base cases if $v_i$ is leaf

Store computed values in $M$ for use by parent of $v_i$.

return $OPT(v_n, 0, 0)$ (* Note: $v_n$ is the root of $T$ *)

**Exercise:** Work out details and prove an $O(n)$ time implementation.
Recap

- To obtain recursive solution we introduced additional variables based on “information” needed to decompose.
- Decomposition depends both on structure (trees decompose via separators) and objective function.
- Subproblems and recursion are almost defined hand in hand.