The Fundamental Theorem of Linear Programming. A canonical linear program $\Pi$ has an optimal solution $x^*$ if and only if the dual linear program $\Pi^*$ has an optimal solution $y^*$ such that $c \cdot x^* = y^* A x^* = y^* \cdot b$.

Weak duality: If $x$ is feasible for $\Pi$, $y$ is feasible for $\Pi^*$, then $c \cdot x \leq y^* A x \leq y^* \cdot b$

Proof: $x$ is feasible $\Rightarrow A x \leq b$ $\Rightarrow y A x \leq y^* \cdot b$

$y$ is feasible $\Rightarrow y \geq 0$ $\Rightarrow y A x \leq y^* \cdot b$

Symmetry $\Rightarrow c \cdot x \leq y^* A x$

Strong duality $\Rightarrow$ no gap
Dual variables $y^*$

Normals to opt. constraints define a coordinate frame.

Dual variables are coeffs of $c$ in this coord frame.

$y^*_x = \text{force applied by constraint 1}$

$y^*_z = \text{force applied by constraint 2}$
\[ \text{basis} = \text{set of } d \text{ linearly independent constraints} \]

(ignores degeneracies)

location \( \rightarrow \) solve equations
value of a basis = c \cdot \text{location}

There are exactly \( (n+d) \) bases

Basis is feasible if \( Ax \leq b \)
where \( x = \text{location} \)
\( \Rightarrow \) vertex of feasible polyhedron

Basis is locally optimal
\( \Rightarrow \) location is optimal for LP with same obj and only constraints in the basis

\[ \begin{align*}
\min & \quad c \cdot y \\
\text{s.t.} & \quad y \cdot A \geq c \\
& \quad y \geq 0
\end{align*} \]

\[ \begin{align*}
\max & \quad c \cdot x \\
\text{s.t.} & \quad A \cdot x \leq b \\
& \quad x \geq 0
\end{align*} \]

\( d \) variables
\( > n+d \) constraints

\( n \) vars
\( > d+n \) constraints

Feasible \( \Rightarrow \) vertices of feasible region

Locally Optimal \( \Rightarrow \) dual feasible

\[ \text{basis} \leftrightarrow \text{dual basis} \]

\( (n+d) = (d+n) \)

\[ \begin{align*}
\text{Feasible} & \leftrightarrow \loc \text{ opt.} \\
\loc \text{ opt.} & \leftrightarrow \text{feasible} \\
\text{Optimal} & \leftrightarrow \text{optimal}
\end{align*} \]
**PrimalSimplex(H):**

if $\cap H = \emptyset$

    return **Infeasible**

$x \leftarrow$ any feasible vertex / basis

while $x$ is not locally optimal

    if every feasible neighbor of $x$ is higher than $x$
        return **Unbounded**
    else
        $x \leftarrow$ any feasible neighbor of $x$ that is lower than $x$

Feasibility — depends on $b$ but not $c$

Local optimality — depends on $c$ but not $b$
**DualSimplex**($H$):

- If there is no locally optimal vertex, return `UNBOUNDED`.
- Set $x ←$ any locally optimal vertex (basis).

  While $x$ is not feasible:

  - **(pivot upward, maintaining local optimality)**
    - If every locally optimal neighbor of $x$ is lower than $x$, return `INFEASIBLE`.
    - Else, set $x ←$ any locally optimal neighbor of $x$ that is higher than $x$.

  Return $x$.

**Note:**

- Two bases are neighbors if and only if they share $d-1$ constraints.
Dual-Primal Simplex ($H$):

1. \( x \leftarrow \text{any vertex} \)
2. \( \hat{H} \leftarrow \text{any rotation of } H \text{ that makes } x \text{ locally optimal} \)

   \[
   \text{Fast while } x \text{ is not feasible }
   \begin{align*}
   &\quad \text{if every locally optimal neighbor of } x \text{ is lower (wrt } \hat{H}) \text{ than } x \\
   &\quad \quad \text{return INFEASIBLE} \\
   &\quad \text{else} \\
   &\quad \quad x \leftarrow \text{any locally optimal neighbor of } x \text{ that is higher (wrt } \hat{H}) \text{ than } x
   \end{align*}

   \text{while } x \text{ is not locally optimal }

   \begin{align*}
   &\quad \text{if every feasible neighbor of } x \text{ is higher than } x \\
   &\quad \quad \text{return UNBOUNDED} \\
   &\quad \text{else} \\
   &\quad \quad x \leftarrow \text{any feasible neighbor of } x \text{ that is lower than } x
   \end{align*}

   return \( x \)
Pick any vertex $x$

Change offsets $b$ so that $x$ is feasible

Pivot to a local opt

Pivot up to a feasible vertex $x^*$

$\text{PrimalDualSimplex}(H)$:

$x \leftarrow$ any vertex

$\tilde{H} \leftarrow$ any translation of $H$ that makes $x$ feasible \text{easy}

while $x$ is not locally optimal

if every feasible neighbor of $x$ is higher (wrt $\tilde{H}$) than $x$

return UNBOUNDED

else

$x \leftarrow$ any feasible neighbor of $x$ that is lower (wrt $\tilde{H}$) than $x$

while $x$ is not feasible

if every locally optimal neighbor of $x$ is lower than $x$

return INFEASIBLE

else

$x \leftarrow$ any locally-optimal neighbor of $x$ that is higher than $x$

return $x$
Brute force enumeration: \( \binom{n^d + d}{d} \) bases
\( \Theta(n^d) \) exponential 😞
Worst case: \( n = d \) \( \binom{2n}{n} \) bases
\( \approx \Theta(4^n \sqrt{n}) \)
Feasible bases: \( \Theta(n^{\frac{d}{2} + 1}) \) \( \mathcal{O}(n) \) when \( d = 2 \cdot 13 \)
\( \mathcal{O}(n^2) \) worst case \( n = d \)

Stupid pivots \( \rightarrow \) exptime
Smart pivots \( \rightarrow \)

best known subexponential super-polynomial # pivots
2d facets in d dim
2d bases
Klee-Minc cubes

\textbf{BIG OPEN QUESTION: Pivot \rightarrow poly # pivots?}

Mostly Fast
Random LP \( \rightarrow \) simplex fast on average
Arbitrary LP + noise \( \rightarrow \) fast on average

"Smoothed analysis" \( \mathcal{O}(n + \log^2 n) \)

Big open question — diameter of feasible polytope
Hirsch conjecture: \( \leq n \) NOPE
Weak \( \leq \mathcal{O}(n) \) OPEN

poly time?
Ellipsoid