Standard flow network:

Directed graph $G = (V, E)$
Capacity $c(e) \geq 0$ for each $e \in E$

Two vertices $s, t \in V$

Feasible Flow =

$F : E \to \mathbb{R}$
$s.t. F(e) \leq c(e)$

$\sum w \in \omega (v \rightarrow w) - \sum u \in u (u \rightarrow v) = b(v)$
for all vertices $v \neq s, t$

Maximize

$\mid F \mid = \sum w \in \omega F(s \rightarrow w) - \sum u \in u F(u \rightarrow s)$

Find a feasible flow if one exists

$\sum b(v) = 0 \Rightarrow \text{demand consumption}$
$b(v) < 0 \Rightarrow \text{supply production}$

F is Feasible $\Rightarrow \sum v \in v (\sum u \in u (u \rightarrow v) - \sum w \in w (v \rightarrow w))$

$= \sum u \in u F(u \rightarrow v) - \sum w \in w F(v \rightarrow w) = 0$

$\sum b(v) = 0$ is necessary but not sufficient

no Feasible Flow because $\sum b(v) \neq 0$
Solve by reducing to standard max flow


Given $G = (V, E)$
$c(e)$ for all $e \in E$
$b(v)$ for all $v \in V$

Construct $G' = (V', E')$
$V' = V \cup \{s, t\}$
$E' = E \cup \{s \rightarrow v \mid b(v) < 0\}$
$E' = E \cup \{v \rightarrow t \mid b(v) > 0\}$
$c(s \rightarrow v) = -b(v)$
$c(v \rightarrow t) = b(v)$

A feasible flow in $G$ and the corresponding saturating flow in $G'$.

Let $F$ be any feasible flow in $G$
Define $F'(u \rightarrow v) = \begin{cases} F(u \rightarrow v) & \text{if } u \rightarrow v \in E \\ -b(v) & \text{if } u = s \\ b(v) & \text{if } v = t \end{cases}$

$f'$ saturating $\Rightarrow F'_{e_{in}} = f$ is feasible

Max flow via Orlin's
$O(|V'|E') = O(|V|E)$ time
Max-flow min-cut theorem? For any $x$,
- Either $|F| \geq x$
- Or $|S,T| \leq x$

In a network with non-zero $b(v)$,
- a cut $(S,T)$ is infeasible if
  \[
  |S,T| = \sum_{u \in S} \sum_{v \in T} c(u \to v) < \sum_{v \in T} b(v) = b(T)
  \]

\[\text{demand of } T\]

Every flow network has either feasible flow or infeasible cut.

Ford-Fulkerson?
Augmenting path?
Residual graph?

\[\text{pseudoFlow: } F: E \to \mathbb{R}\]
- Feasible: $0 \leq F(e) \leq c(e)$ for all $e$
- Balanced: $\sum_{u \in S} F(u \to v) - \sum_{u \in T} F(u \to v) = b(v)$ for all $v$
- Flow = balanced pseudoflow

From left to right: A pseudoflow $\psi$ in a flow network $G$; the residual graph $G_{\psi}$ with one augmenting path highlighted; and the updated pseudoflow after pushing 4 units along the augmenting path.

Aug path = path from $u$ to $v$ such that $b_{\psi}(u) > 0$ and $b_{\psi}(d) > 0$. 
\[ b_f(v) = b(v) - \left( \sum_u f(u \rightarrow v) - \sum_w f(v \rightarrow w) \right) \]

**Feasible Flow** \((V, E, c, b)\):

- for every edge \( e \in E \)
  
  \[ \psi(e) \leftarrow 0 \]

  \[ B \leftarrow \sum_v |b(v)|/2 \]

  while \( B > 0 \)
  
  construct \( G_\psi \)
  
  \( \langle \text{Find augmenting path } \pi \rangle \)
  
  \( s \leftarrow \text{any vertex with } b_\psi(s) < 0 \)
  
  if \( s \) cannot reach a vertex \( t \) in \( G_\psi \) with \( b_\psi(t) > 0 \)
  
  return **INFEASIBLE**

  \( t \leftarrow \text{any vertex reachable from } s \) with \( b_\psi(t) > 0 \)

  \( \pi \leftarrow \text{any path in } G_\psi \text{ from } s \) to \( t \)

  \( \langle \text{Push as much flow as possible along } \pi \rangle \)
  
  \[ R \leftarrow \min \{-b_\psi(s), b_\psi(t), \min_{e \in \pi} c_\psi(e)\} \]

  \[ B \leftarrow B - R \]

  for every directed edge \( e \in \pi \)
  
  if \( e \in E \)
  
  \[ \psi(e) \leftarrow \psi(e) + R \]

  else \( \langle \text{rev}(e) \in E \rangle \)
  
  \[ \psi(e) \leftarrow \psi(e) - R \]

  return \( \psi \)

**Integer** \( \Rightarrow O(EB) \) time

**Orlin’s** \( \Rightarrow O(EV) \) time
Maximum Flow with non-zero balances

1. Feasible (balanced) flow $F$ in $G$
2. Find max flow $F'$ in $G_f$ — standard
3. Return $F + F'$
   value = $|F| + |F'|$

Feasible flow $F$ might have non-zero value

Feasible (balanced) flow

$O(VE)$ time
Max-flow with lower bounds on the edges

1. Find feasible flow $F_0$ in $G$
2. Find max-flow $f'$ in $G_f$
3. Return $f + f'$

$f(u \rightarrow v) = -f(v \rightarrow u)$
$c(u \rightarrow v) = -l(v \rightarrow u)$
$l(u \rightarrow v) = -c(v \rightarrow u)$

1(a) Find a feasible pseudoflow $f_0$ in $G$

(b) Find a feasible balanced flow $F_1$ in $G_f$

2. Find max (feasible, balanced) flow in $G_{f_1}$

Max-flow in $G_{f_0}$

$0 \leq F \leq c$

$0 \leq l - F \leq 0 \leq c - F$

$c - F$

$f - l$

$O(VE)$ time