Maximum Flows and Minimum Cuts

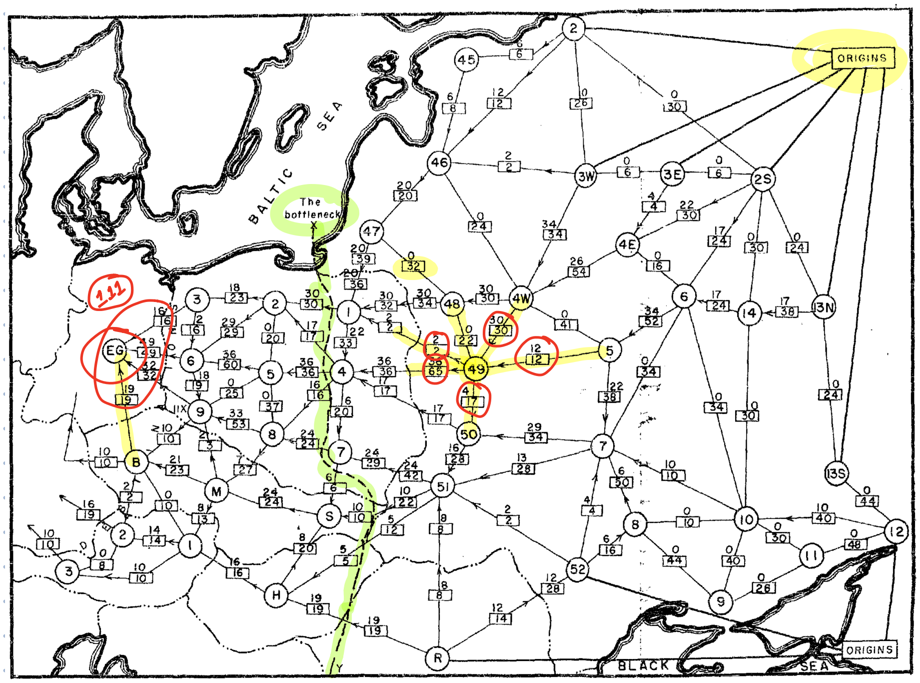

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Fig. 7 — Traffic pattern: entire network available

Legend:

International boundary

Railway operating division

Capacity: 12 each way per day. Required flow of 9 per day toward destinations (in direction of arrow) with equivalent number of returning trains in opposite direction

All capacities in trains / 1000's of tons each way per day

Origins: Divisions 2, 3W, 3E, 2S, 13N, 13S, 12, 52 (USSR), and Roumania

Destinations: Divisions 3, 6, 9 (Poland); B (Czechoslovakia); and 2, 3 (Austria)

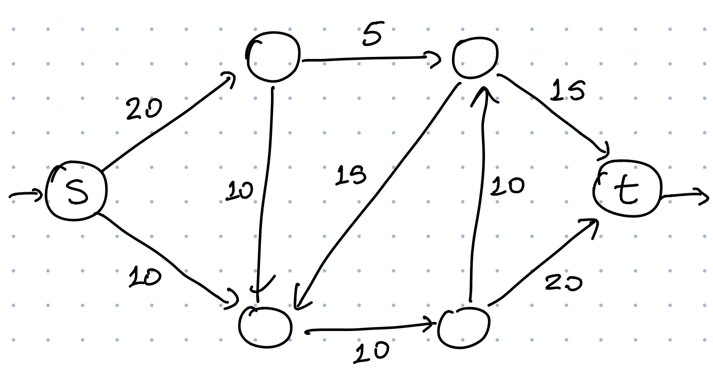
Alternative destinations: Germany or East Germany

Note IIX at Division 9, Poland

Flow network

- Directed graph G

- source vertex s
- target vertex t
- capacities for edges c(e)



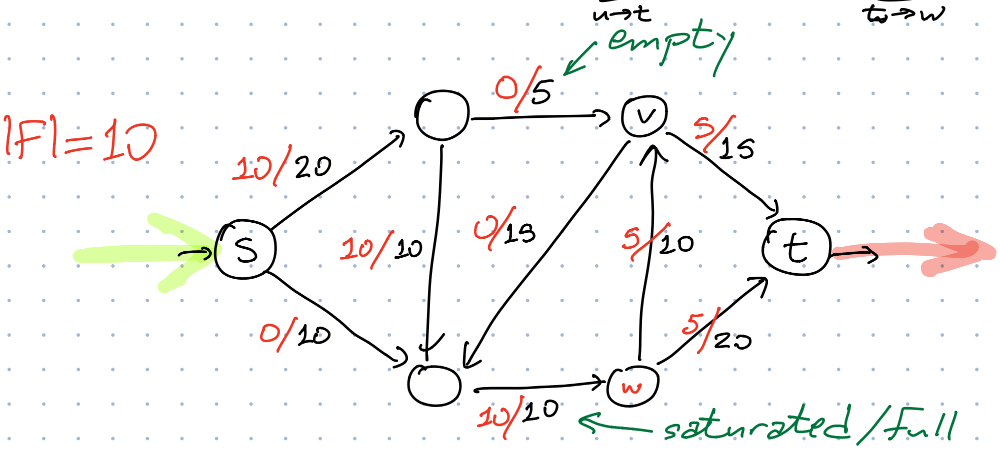
(s,t)-Flow — Function $f: E \to \mathbb{R}$ $\quad f(e) = $ Flow thru e

- conservation:

$$\sum_{u \to v} f(u\to v) = \sum_{v \to w} f(v \to w) \quad \text{for all } v \neq s,t$$

- Feasible: $0 \le f(u\to v) \le c(u \to v)$ for all $u \to v$
- Maximize value $|f| = \sum_{s\to w} f(s\to w) - \sum_{u\to s} f(u\to s)$

$$= \sum_{u\to t} f(u\to t) - \sum_{t\to w} f(t\to w)$$



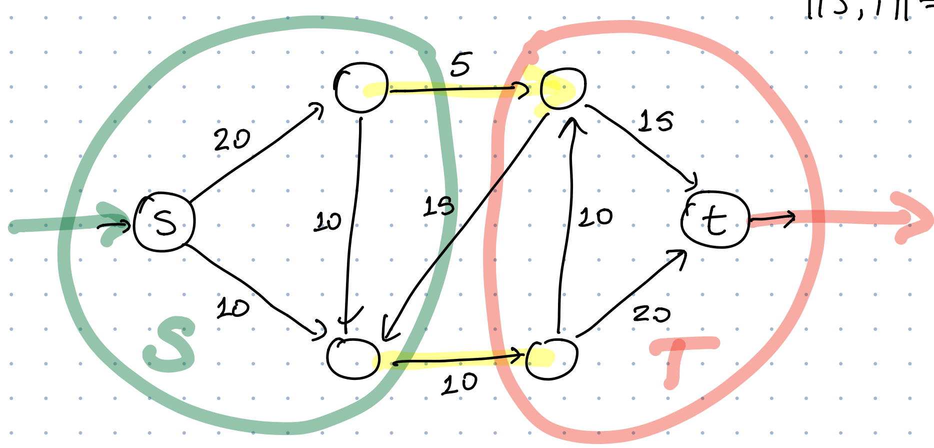
(s,t)-cut = partition $V = S \cup T$ $\quad S \cap T = \emptyset$

where $s \in S$ and $t \in T$

minimize

- capacity $\|S,T\| = \sum_{u\in S} \sum_{v \in T} c(u \to v)$

$\|S,T\| = 15$



Maximum Flow – Minimum Cut Theorem

In any flow network
value of max flow = cap of min cut

Easy direction

For any ~~flow~~ feasible $\underline{f}$ and any cut $\underline{(S,T)}$ in same flow network G

$$|f| \le \|S,T\|$$

$$|f| = \sum_{s\to w} f(s\to w) - \sum_{u\to s} f(u\to s) = \text{net flow out of } s \qquad \text{[definition]}$$

$$= \text{net flow out of } S \text{ into } T \qquad \text{[conservation]}$$

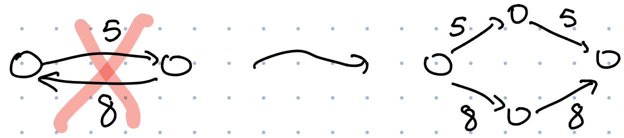
$$= \sum_{u\in S}\sum_{v\in T} f(u\to v) - \sum_{u\in S}\sum_{v\in T} f(v\to u)$$

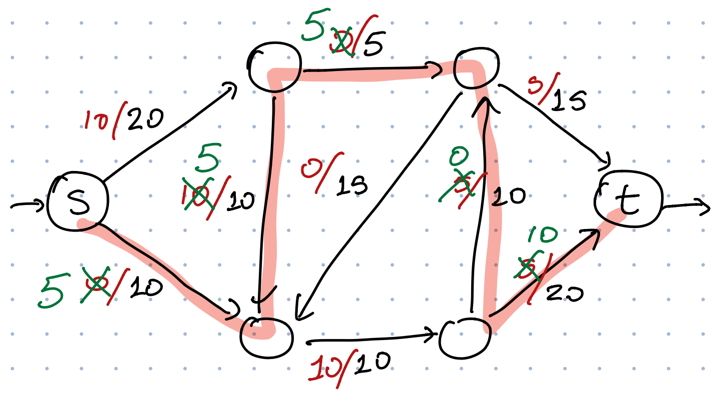
$$\le \sum_{u\in S}\sum_{v\in T} c(u\to v) - \sum_{u\in S}\sum_{v\in T} 0 \qquad \text{[Feasibility]}$$

$$= \|S,T\| \qquad \text{[definition]}$$

If $|f| = \|S,T\|$ then f is a max flow and (S,T) is a min cut.

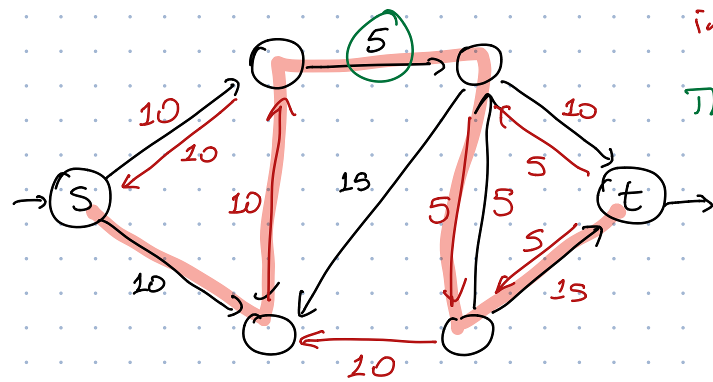
Claim: Max flow $\geq$ min cut.





We can increase $|f|$ by 5

Residual Graph G_f — How much more flow can go thru G in addition to f?

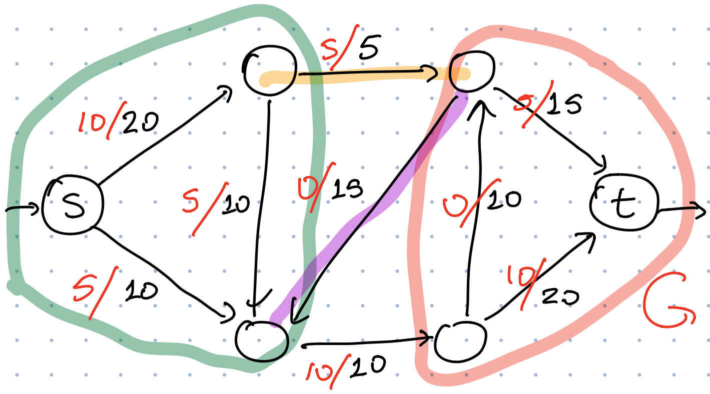


The residual path has capacity 5

Residual capacity $c_f(u \to v) = \begin{cases} c(u \to v) - f(u \to v) & \text{if } u \to v \in E \\ f(v \to u) & \text{if } v \to u \in E \\ 0 & \text{otherwise} \end{cases}$

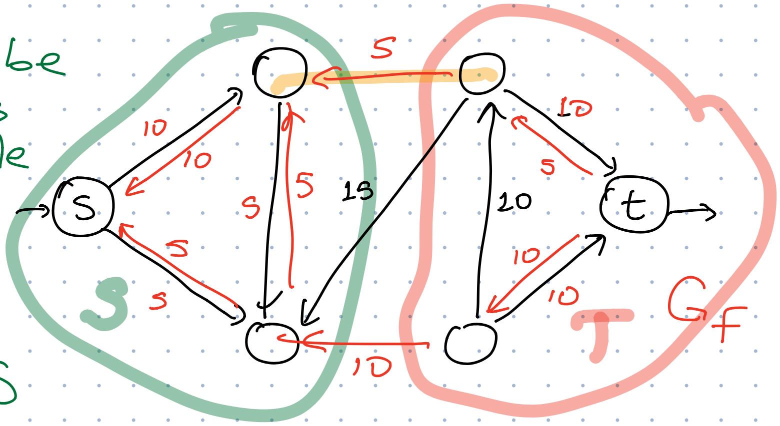
If the residual graph G_f has a path from s to t, then f is NOT a max flow

If G_f has <u>no</u> path from s to t, then...



Let S be vertices reachable from s in G_f

Let $T = V \setminus S$



Pick $u \in S$ and $v \in T$

$u \to v \in E \implies u \to v$ is not in G_f

$c_f(u \to v) = 0$

$f(u \to v) = c(u \to v)$

$v \to u \in E \implies u \to v$ is not in G_f

$c_f(u \to v) = 0$

$f(u \to v) = 0$

Every $S \to T$ edge is full

Every $T \to S$ edge is empty

$\implies |f| = \|S, T\| \implies$ f is max flow, (S, T) is min cut

Ford Fulkerson RAND 1950s

Elias Feinstein Shannon

Algorithm

$f \leftarrow 0$

$G_f \leftarrow G$

while G_f has s→t path

augment f along that path

rebuild G_f

return f

If this algorithm halts, it returns max flow

If all caps are ints then

FF alg halts after $\le |f^*|$ iteration

If allow irrational caps

FF can ∞ loop

doesn't even converge to max flow