- **The World's Most Useful Limit:** $\lim_{n\to\infty}\left(1+\frac{\alpha}{n}\right)^n = e^\alpha$ for any real number α.
- **The World's Most Useful Inequality:** $1 + x \le e^x$ for all x

Tail Inequalities

Suppose X is the sum of random indicator variables $X_1, X_2, \ldots, X_n$.
For each index i, let $p_i = \Pr[X_i = 1] = \mathrm{E}[X_i]$, and let $\mu = \sum_i p_i = \mathrm{E}[X]$.

- **Markov's Inequality:**

$$\Pr[X \ge x] \le \frac{\mu}{x} \qquad \text{for all } x > 0 \text{, and therefore...}$$

$$\Pr[X \ge (1+\delta)\mu] \le \frac{1}{1+\delta} \qquad \text{for all } \delta > 0$$

- **Chebyshev's Inequality:** If the variables X_i are pairwise independent, then...

$$\Pr[(X-\mu)^2 \ge z] < \frac{\mu}{z} \qquad \text{for all } z > 0 \text{, and therefore...}$$

$$\Pr[X \ge (1+\delta)\mu] < \frac{1}{\delta^2\mu} \qquad \text{for all } \delta > 0$$

$$\Pr[X \le (1-\delta)\mu] < \frac{1}{\delta^2\mu} \qquad \text{for all } \delta > 0$$

- **Higher Moment Inequalities:** If the variables X_i are $2k$-wise independent, then...

$$\Pr[(X-\mu)^{2k} \ge z] = O\left(\frac{\mu^k}{z}\right) \qquad \text{for all } z > 0 \text{, and therefore...}$$

$$\Pr[X \ge (1+\delta)\mu] = O\left(\frac{1}{\delta^{2k}\mu^k}\right) \qquad \text{for all } \delta > 0$$

$$\Pr[X \le (1-\delta)\mu] = O\left(\frac{1}{\delta^{2k}\mu^k}\right) \qquad \text{for all } \delta > 0$$

- **Chernoff's Inequality:** If the variables X_i are fully independent, then...

$$\Pr[X \ge x] \le e^{x-\mu}\left(\frac{\mu}{x}\right)^x \qquad \text{for all } x \ge \mu \text{, and therefore...}$$

$$\Pr[X \ge (1+\delta)\mu] \le e^{-\delta^2\mu/3} \qquad \text{for all } 0 < \delta < 1$$

$$\Pr[X \le (1-\delta)\mu] \le e^{-\delta^2\mu/2} \qquad \text{for all } 0 < \delta < 1$$

Hashing Properties

$\mathcal{H}$ is a set of functions from some universe $\mathcal{U}$ to $[m] = \{0, 1, 2, \ldots, m-1\}$.

- **Universal:** $\Pr_{h\in\mathcal{H}}[h(x) = h(y)] \le \frac{1}{m}$ for all distinct items $x \ne y$
- **Near-universal:** $\Pr_{h\in\mathcal{H}}[h(x) = h(y)] \le O\left(\frac{1}{m}\right)$ for all distinct items $x \ne y$
- **Strongly universal:** $\Pr_{h\in\mathcal{H}}[h(x) = i \text{ and } h(y) = j] = \frac{1}{m^2}$ for all distinct $x \ne y$ and all i and j
- **2-uniform:** Same as strongly universal.
- **Ideal Random:** Complete fiction.