CS 473: Algorithms

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Basics of Discrete Probability

Discrete Probability

We restrict attention to finite probability spaces.

Definition

A discrete probability space is a pair $(\Omega, \Pr)$ consists of finite set Ω of **elementary events** and function $p : \Omega \to [0, 1]$ which assigns a probability $\Pr[\omega]$ for each $\omega \in \Omega$ such that $\sum_{\omega \in \Omega} \Pr[\omega] = 1$.

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Example

An unbiased coin. $\Omega = \{H, T\}$ and $\mathbf{Pr}[H] = \mathbf{Pr}[T] = 1/2$.

Example

A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathbf{Pr}[i] = 1/6$ for $1 \leq i \leq 6$.

Discrete Probability

And more examples

Example

A biased coin. $\Omega = \{H, T\}$ and $\Pr[H] = 2/3, \Pr[T] = 1/3$.

Example

Two independent unbiased coins. $\Omega = \{HH, TT, HT, TH\}$ and $\Pr[HH] = \Pr[TT] = \Pr[HT] = \Pr[TH] = 1/4$.

Example

A pair of (highly) correlated dice.
$\Omega = \{(i, j) \mid 1 \leq i \leq 6, 1 \leq j \leq 6\}$.
$\Pr[i, i] = 1/6$ for $1 \leq i \leq 6$ and $\Pr[i, j] = 0$ if $i \neq j$.

Events

Definition

Given a probability space $(\Omega, \Pr)$ an **event** is a subset of Ω. In other words an event is a collection of elementary events. The probability of an event A, denoted by $\Pr[A]$, is $\sum_{\omega \in A} \Pr[\omega]$.

The **complement event** of an event $A \subseteq \Omega$ is the event $\Omega \setminus A$ frequently denoted by $\bar{A}$.

Example

A pair of independent dice. $\Omega = \{(i, j) \mid 1 \leq i \leq 6, 1 \leq j \leq 6\}$.

1. Let A be the event that the sum of the two numbers on the dice is even.
   Then $A = \left\{ (i, j) \in \Omega \,\middle|\, (i + j) \text{ is even} \right\}$.
   $\Pr[A] = |A|/36 = 1/2$.

2. Let B be the event that the first die has 1. Then
   $B = \left\{ (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6) \right\}$.
   $\Pr[B] = 6/36 = 1/6$.

Independent Events

Definition

Given a probability space $(\Omega, \Pr)$ and two events A, B are **independent** if and only if $\Pr[A \cap B] = \Pr[A] \Pr[B]$. Otherwise they are *dependent*. In other words A, B independent implies one does not affect the other.

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Example

Two coins. $\Omega = \{HH, TT, HT, TH\}$ and $\Pr[HH] = \Pr[TT] = \Pr[HT] = \Pr[TH] = 1/4$.

1. A is the event that the first coin is heads and B is the event that second coin is tails. A, B are independent.
2. A is the event that the two coins are different. B is the event that the second coin is heads. A, B independent.

Independent Events

Examples

Example

A is the event that both are not tails and **B** is event that second coin is heads. **A, B** are dependent.

Dependent or independent?

Consider two independent rolls of the dice.

1. $\mathbf{A} =$ the event that the first roll is odd.
2. $\mathbf{B} =$ the event that the sum of the two rolls is odd.

The events $\mathbf{A}$ and $\mathbf{B}$ are

(A) dependent.

(B) independent.

Union bound

The probability of the union of two events, is no bigger than the probability of the sum of their probabilities.

Lemma

For any two events $\mathcal{E}$ *and* $\mathfrak{F}$*, we have that*
$\mathbf{Pr}\left[\mathcal{E} \cup \mathfrak{F}\right] \leq \mathbf{Pr}\left[\mathcal{E}\right] + \mathbf{Pr}\left[\mathfrak{F}\right]$.

Proof.

Consider $\mathcal{E}$ and $\mathfrak{F}$ to be a collection of elmentery events (which they are). We have

$$\begin{aligned} \mathbf{Pr}\left[\mathcal{E} \cup \mathfrak{F}\right] &= \sum_{x \in \mathcal{E} \cup \mathfrak{F}} \mathbf{Pr}[x] \\ &\leq \sum_{x \in \mathcal{E}} \mathbf{Pr}[x] + \sum_{x \in \mathfrak{F}} \mathbf{Pr}[x] = \mathbf{Pr}\left[\mathcal{E}\right] + \mathbf{Pr}\left[\mathfrak{F}\right]. \end{aligned}$$

Random Variables

Definition

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1. $X : \Omega \to \mathbb{R}$ where $X(i) = i \mod 2$.
2. $Y : \Omega \to \mathbb{R}$ where $Y(i) = i^2$.

Expectation

Definition

For a random variable $\mathbf{X}$ over a probability space $(\boldsymbol{\Omega}, \mathbf{Pr})$ the **expectation** of $\mathbf{X}$ is defined as $\sum_{\omega \in \Omega} \mathbf{Pr}[\omega]\, \mathbf{X}(\omega)$. In other words, the expectation is the average value of $\mathbf{X}$ according to the probabilities given by $\mathbf{Pr}[\cdot]$.

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Example

A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Pr[i] = 1/6$ for $1 \leq i \leq 6$.

1. $X : \Omega \to \mathbb{R}$ where $X(i) = i \mod 2$. Then $E[X] = 1/2$.
2. $Y : \Omega \to \mathbb{R}$ where $Y(i) = i^2$. Then $E[Y] = \sum_{i=1}^{6} \frac{1}{6} \cdot i^2 = 91/6$.

Expected number of vertices?

Let $G = (V, E)$ be a graph with n vertices and m edges. Let H be the graph resulting from independently deleting every vertex of G with probability $1/2$. The expected number of vertices in H is

- **(A)** $n/2$.
- **(B)** $n/4$.
- **(C)** $m/2$.
- **(D)** $m/4$.
- **(E)** none of the above.

Expected number of edges?

Let $G = (V, E)$ be a graph with $\mathbf{n}$ vertices and $\mathbf{m}$ edges. Let H be the graph resulting from independently deleting every vertex of G with probability $\mathbf{1/2}$. The expected number of edges in H is

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Special type of random variables that are quite useful.

Definition

Given a probability space $(\Omega, \Pr)$ and an event $A \subseteq \Omega$ the indicator random variable X_A is a binary random variable where $X_A(\omega) = 1$ if $\omega \in A$ and $X_A(\omega) = 0$ if $\omega \notin A$.

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Example

A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Pr[i] = 1/6$ for $1 \leq i \leq 6$. Let A be the even that i is divisible by 3. Then $X_A(i) = 1$ if $i = 3, 6$ and 0 otherwise.

Expectation

Proposition

For an indicator variable X_A, $E[X_A] = Pr[A]$.

Proof.

$$\begin{aligned} E[X_A] &= \sum_{y \in \Omega} X_A(y) \Pr[y] \\ &= \sum_{y \in A} 1 \cdot \Pr[y] + \sum_{y \in \Omega \setminus A} 0 \cdot \Pr[y] \\ &= \sum_{y \in A} \Pr[y] \\ &= \Pr[A]. \end{aligned}$$

Linearity of Expectation

Lemma

Let $\mathbf{X}, \mathbf{Y}$ *be two random variables (not necessarily independent) over a probability space* $(\Omega, \Pr)$. *Then* $\mathbf{E}[\mathbf{X} + \mathbf{Y}] = \mathbf{E}[\mathbf{X}] + \mathbf{E}[\mathbf{Y}]$.

Proof.

$$\begin{aligned}
\mathbf{E}[\mathbf{X} + \mathbf{Y}] &= \sum_{\omega \in \Omega} \Pr[\omega] \left(\mathbf{X}(\omega) + \mathbf{Y}(\omega)\right) \\
&= \sum_{\omega \in \Omega} \Pr[\omega] \, \mathbf{X}(\omega) + \sum_{\omega \in \Omega} \Pr[\omega] \, \mathbf{Y}(\omega) = \mathbf{E}[\mathbf{X}] + \mathbf{E}[\mathbf{Y}] \, .
\end{aligned}$$

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Proof.

$$\mathbf{E}[\mathbf{X} + \mathbf{Y}] = \sum_{\omega \in \Omega} \Pr[\omega] \left(\mathbf{X}(\omega) + \mathbf{Y}(\omega)\right)$$

$$= \sum_{\omega \in \Omega} \Pr[\omega] \, \mathbf{X}(\omega) + \sum_{\omega \in \Omega} \Pr[\omega] \, \mathbf{Y}(\omega) = \mathbf{E}[\mathbf{X}] + \mathbf{E}[\mathbf{Y}] \, .$$

□

Corollary

$\mathbf{E}[a_1 \mathbf{X}_1 + a_2 \mathbf{X}_2 + \ldots + a_n \mathbf{X}_n] = \sum_{i=1}^{n} a_i \, \mathbf{E}[\mathbf{X}_i]$.

Expected number of edges?

Let $G = (V, E)$ be a graph with $\mathbf{n}$ vertices and $\mathbf{m}$ edges. Let H be the graph resulting from independently deleting every vertex of G with probability $\mathbf{1/2}$. The expected number of edges in H is

- **(A)** $n/2$.
- **(B)** $n/4$.
- **(C)** $m/2$.
- **(D)** $m/4$.
- **(E)** none of the above.

Expected number of triangles?

Let $G = (V, E)$ be a graph with $\mathbf{n}$ vertices and $\mathbf{m}$ edges. Assume G has $\mathbf{t}$ triangles (i.e., a triangle is a simple cycle with three vertices). Let H be the graph resulting from deleting independently each vertex of G with probability $1/2$. The expected number of triangles in H is

- **(A)** $t/2$.
- **(B)** $t/4$.
- **(C)** $t/8$.
- **(D)** $t/16$.
- **(E)** none of the above.