CS 473: Algorithms

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Simplex and LP Duality

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Some of the slides are courtesy Prof. Chekuri
Outline

Simplex: Intuition and Implementation Details
  - Computing starting vertex: equivalent to solving an LP!

Infeasibility, Unboundedness, and Degeneracy.

Duality: Bounding the objective value through \textit{weak-duality}

Strong Duality, Cone view.
Part I

Recall
Feasible Region and Convexity

Canonical Form

Given $A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^{n \times 1}$ and $c \in \mathbb{R}^{1 \times d}$, find $x \in \mathbb{R}^{d \times 1}$

\[
\begin{align*}
\text{max :} & \quad c \cdot x \\
\text{s.t.} & \quad Ax \leq b
\end{align*}
\]
Feasible Region and Convexity

Canonical Form

Given \( A \in \mathbb{R}^{n \times d}, b \in \mathbb{R}^{n \times 1} \) and \( c \in \mathbb{R}^{1 \times d} \), find \( x \in \mathbb{R}^{d \times 1} \)

\[
\begin{align*}
\max : & \quad c \cdot x \\
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\end{align*}
\]

1. Each linear constraint defines a **halfspace**, a convex set.
2. Feasible region, which is an intersection of halfspaces, is a convex **polyhedron**.
3. Optimal value attained at a vertex of the polyhedron.
Simplex Algorithm

Simplex: Vertex hoping algorithm

Moves from a vertex to its neighboring vertex
Simplex Algorithm

Simplex: Vertex hoping algorithm

 Moves from a vertex to its neighboring vertex

Questions

- Which neighbor to move to?
- When to stop?
- How much time does it take?
Suppose we are at a non-optimal vertex $\hat{x}$ and optimal is $x^*$, then $c \cdot x^* > c \cdot \hat{x}$.
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How does $(c \cdot x)$ change as we move from $\hat{x}$ to $x^*$ on the line joining the two?
Suppose we are at a non-optimal vertex $\hat{x}$ and optimal is $x^*$, then $c \cdot x^* > c \cdot \hat{x}$.

How does $(c \cdot x)$ change as we move from $\hat{x}$ to $x^*$ on the line joining the two?

Strictly increases!
Given a set of vectors $D = \{d_1, \ldots, d_k\}$, the cone spanned by them is just their positive linear combinations, i.e.,

$$cone(D) = \{d \mid d = \sum_{i=1}^{k} \lambda_i d_i, \text{ where } \lambda_i \geq 0, \forall i\}$$
Cone at a Vertex

Let $z_1, \ldots, z_k$ be the neighboring vertices of $\hat{x}$. And let $d_i = z_i - \hat{x}$ be the direction from $\hat{x}$ to $z_i$.

**Lemma**

Any feasible direction of movement $d$ from $\hat{x}$ is in the cone($\{d_1, \ldots, d_k\}$).
Improving Direction Implies Improving Neighbor

**Lemma**

If $d \in \text{cone}(\{d_1, \ldots, d_k\})$ and $(c \cdot d) > 0$, then there exists $d_i$ such that $(c \cdot d_i) > 0$.

**Proof.** To the contrary suppose $(c \cdot d_i) \leq 0$, $\forall i \leq k$.

Since $d$ is a positive linear combination of $d_i$'s,

$(c \cdot d) = (c \cdot \sum_{i=1}^{k} \lambda_i d_i) = \sum_{i=1}^{k} \lambda_i (c \cdot d_i) \leq 0$

A contradiction!

**Theorem**

If vertex $\hat{x}$ is not optimal then it has a neighbor where cost improves.
Lemma

If \( d \in \text{cone}\{d_1, \ldots, d_k\} \) and \((c \cdot d) > 0\), then there exists \( d_i \) such that \((c \cdot d_i) > 0\).

Proof.

To the contrary suppose \((c \cdot d_i) \leq 0\), \( \forall i \leq k \).

Since \( d \) is a positive linear combination of \( d_i \)'s,

\[
(c \cdot d) = (c \cdot \sum_{i=1}^{k} \lambda_i d_i) \\
= \sum_{i=1}^{k} \lambda_i (c \cdot d_i) \\
\leq 0 \quad \text{A contradiction!}
\]
Lemma

If \( d \in \text{cone}(\{d_1, \ldots, d_k\}) \) and \((c \cdot d) > 0\), then there exists \( d_i \) such that \((c \cdot d_i) > 0\).

Proof.

To the contrary suppose \((c \cdot d_i) \leq 0\), \( \forall i \leq k \).
Since \( d \) is a positive linear combination of \( d_i \)'s,

\[
(c \cdot d) = (c \cdot \sum_{i=1}^{k} \lambda_i d_i) = \sum_{i=1}^{k} \lambda_i (c \cdot d_i) \leq 0 \quad \text{A contradiction!}
\]

Theorem

If vertex \( \hat{x} \) is not optimal then it has a neighbor where cost improves.
How Many Neighbors a Vertex Has?

Geometric view...

\[ A \in \mathbb{R}^{n \times d} \ (n > d), \ b \in \mathbb{R}^n, \ \text{the constraints are:} \ Ax \leq b \]

**Geometry of faces**

- \( r \) linearly independent hyperplanes forms \((d - r)\) dimensional face.
How Many Neighbors a Vertex Has?

Geometric view...

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- Vertex: \( 0 \)-D face. formed by \( d \) L.I. hyperplanes.
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- Edge: \(1\)-D face. formed by \((d - 1)\) L.I. hyperlanes.
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Geometric view...

\[ A \in \mathbb{R}^{n \times d} \ (n > d), \ b \in \mathbb{R}^n, \] the constraints are: \[ Ax \leq b \]

**Geometry of faces**

- \( r \) linearly independent hyperplanes forms \((d - r)\) dimensional face.
- Vertex: **0**-D face. formed by \( d \) L.I. hyperplanes.
- Edge: **1**-D face. formed by \((d - 1)\) L.I. hyperplanes.

In 2-dimension \((d = 2)\)
How Many Neighbors a Vertex Has?

Geometric view...

In 3-dimension \((d = 3)\)

\(A \in R^{n \times d} \ (n > d), \ b \in R^n\), the constraints are: \(Ax \leq b\)

Geometry of faces

- \(r\) linearly independent hyperplanes forms \((d - r)\) dimensional face.
- Vertex: 0-dimensional face. formed by \(d\) L.I. hyperplanes.
- Edge: 1-D face. formed by \((d - 1)\) L.I. hyperplanes.

image source: webpage of Prof. Forbes W. Lewis
How Many Neighbors a Vertex Has?

Geometry view...

One neighbor per tight hyperplane. Therefore typically $d$.

- Suppose $x'$ is a neighbor of $\hat{x}$, then on the edge joining the two $d - 1$ constraints are tight.
- These $d - 1$ are also tight at both $\hat{x}$ and $x'$.
- One more constraints, say $i$, is tight at $\hat{x}$. “Relaxing” $i$ at $\hat{x}$ leads to $x'$.
Simplex Algorithm

Simplex: Vertex hoping algorithm

Moves from a vertex to its neighboring vertex

Questions + Answers

- Which neighbor to move to? One where objective value increases.
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Questions + Answers

- Which neighbor to move to? One where objective value increases.
- When to stop? When no neighbor with better objective value.
Simplex Algorithm

Simplex: Vertex hoping algorithm

Moves from a vertex to its neighboring vertex

Questions + Answers

- Which neighbor to move to? One where objective value increases.
- When to stop? When no neighbor with better objective value.
- How much time does it take? At most $d$ neighbors to consider in each step.
Simplex in Higher Dimensions

**Simplex Algorithm**

1. Start at a vertex of the polytope.
2. Compare value of objective function at each of the $d$ “neighbors”.
3. Move to neighbor that improves objective function, and repeat step 2.
4. If no improving neighbor, then stop.

Simplex is a greedy local-improvement algorithm! Works because a local optimum is also a global optimum — convexity of polyhedra.
Naive implementation of Simplex algorithm can be very inefficient – Exponential number of steps!
Naïve implementation of Simplex algorithm can be very inefficient

1. Choosing which neighbor to move to can significantly affect running time
2. Very efficient Simplex-based algorithms exist
3. Simplex algorithm takes exponential time in the worst case but works extremely well in practice with many improvements over the years

Non Simplex based methods like interior point methods work well for large problems.
Major open problem for many years: is there a polynomial time algorithm for linear programming?
Polynomial time Algorithm for Linear Programming

Major open problem for many years: is there a polynomial time algorithm for linear programming?
Leonid Khachiyan in 1979 gave the first polynomial time algorithm using the Ellipsoid method.

1. major theoretical advance
2. highly impractical algorithm, not used at all in practice
3. routinely used in theoretical proofs.
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Following interior point method success, Simplex has been improved enormously and is the method of choice.
Issues

1. Starting vertex
2. The linear program could be infeasible: No point satisfy the constraints.
3. The linear program could be unbounded: Polygon unbounded in the direction of the objective function.
4. More than $d$ hyperplanes could be tight at a vertex, forming more than $d$ neighbors.
Computing the Starting Vertex
Equivalent to solving another LP!

Find an \( x \) such that \( Ax \leq b \).
If \( b \geq 0 \) then trivial!
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\[
\begin{align*}
\min : & \quad s \\
\text{s.t.} : & \quad \sum_j a_{ij} x_j - s \leq b_i, \quad \forall i \\
& \quad s \geq 0
\end{align*}
\]

Trivial feasible solution:
Computing the Starting Vertex
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$$s \geq 0$$

Trivial feasible solution: $x = 0, s = |\min_i b_i|$.

If $Ax \leq b$ feasible then optimal value of the above LP is $s = 0$. 
Computing the Starting Vertex

Equivalent to solving another LP!

Find an \( x \) such that \( Ax \leq b \).
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Trivial feasible solution: \( x = 0, \ s = |\min_i b_i| \).

If \( Ax \leq b \) feasible then optimal value of the above LP is \( s = 0 \).

Checks Feasibility!
Unboundedness depends on both constraints and the objective function.
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If unbounded in the direction of objective function, then the pivoting step in the simplex will detect it.
Degeneracy and Cycling

More than $d$ constraints are tight at vertex $\hat{x}$. Say $d + 1$.

Suppose, we pick first $d$ to form $\hat{A}$ such that $\hat{A}\hat{x} = \hat{b}$, and compute directions $d_1, \ldots, d_d$. Then $\text{NextVertex}(\hat{x}, d_i)$ will encounter $(d + 1)$th constraint tight at $\hat{x}$ and return the same vertex. Hence we are back to $\hat{x}$.

This can be avoided by adding small random perturbation to $b_i$s.
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Same phenomenon will repeat!
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Same phenomenon will repeat!

This can be avoided by adding small random perturbation to $b_i$s.
Consider the program

\[
\begin{align*}
\text{maximize} & \quad 4x_1 + 2x_2 \\
\text{subject to} & \quad x_1 + 3x_2 \leq 5 \\
& \quad 2x_1 - 4x_2 \leq 10 \\
& \quad x_1 + x_2 \leq 7 \\
& \quad x_1 \leq 5
\end{align*}
\]
Consider the program

maximize $4x_1 + 2x_2$
subject to $x_1 + 3x_2 \leq 5$
$2x_1 - 4x_2 \leq 10$
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$(0, 1)$ satisfies all the constraints and gives value 2 for the objective function.
Consider the program

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1. \((0, 1)\) satisfies all the constraints and gives value 2 for the objective function.

2. Thus, optimal value \(\sigma^*\) is at least 4.
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3. \((2, 0)\) also feasible, and gives a better bound of \(8\).
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2. Thus, optimal value \(\sigma^*\) is at least 4.

3. \((2, 0)\) also feasible, and gives a better bound of 8.

4. How good is 8 when compared with \(\sigma^*\)?
Obtaining Upper Bounds

maximize $4x_1 + 2x_2$
subject to $x_1 + 3x_2 \leq 5$
$2x_1 - 4x_2 \leq 10$
$x_1 + x_2 \leq 7$
$x_1 \leq 5$

Let us multiply the first constraint by 2 and the second constraint.
Obtaining Upper Bounds

maximize \[ 4x_1 + 2x_2 \]
subject to \[ \begin{align*}
    x_1 + 3x_2 & \leq 5 \\
    2x_1 - 4x_2 & \leq 10 \\
    x_1 + x_2 & \leq 7 \\
    x_1 & \leq 5
\end{align*} \]

Let us multiply the first constraint by 2 and the second constraint and add it to second constraint

\[
\begin{align*}
    2(x_1 + 3x_2) & \leq 2(5) \\
    +1(2x_1 - 4x_2) & \leq 1(10) \\
    \hline
    4x_1 + 2x_2 & \leq 20
\end{align*}
\]
Obtaining Upper Bounds

maximize \[ 4x_1 + 2x_2 \]
subject to \[ x_1 + 3x_2 \leq 5 \]
    \[ 2x_1 - 4x_2 \leq 10 \]
    \[ x_1 + x_2 \leq 7 \]
    \[ x_1 \leq 5 \]

1. Let us multiply the first constraint by 2 and the and add it to second constraint

\[
2(x_1 + 3x_2) \leq 2(5) \\
+1(2x_1 - 4x_2) \leq 1(10) \\
\overline{4x_1 + 2x_2} \leq 20
\]

2. Thus, 20 is an upper bound on the optimum value!
Multiply first equation by \( y_1 \), second by \( y_2 \), third by \( y_3 \) and fourth by \( y_4 \) \( (y_1, y_2, y_3, y_4 \geq 0) \) and add

\[
\begin{align*}
&y_1(x_1 + \ldots + 3x_2) \leq y_1(5) \\
&+ y_2(2x_1 - \ldots + 4x_2) \leq y_2(10) \\
&+ y_3(x_1 + \ldots + x_2) \leq y_3(7) \\
&+ y_4(x_1 + \ldots + x_1) \leq y_4(5) \\
&(y_1 + 2y_2 + y_3 + y_4)x_1 + (3y_1 - 4y_2 + y_3)x_2 \leq \ldots
\end{align*}
\]
Generalizing...

1. Multiply first equation by $y_1$, second by $y_2$, third by $y_3$ and fourth by $y_4$ ($y_1, y_2, y_3, y_4 \geq 0$) and add

\[
\begin{align*}
y_1(x_1 + 2x_2 - 4x_2) & \leq y_1(5) \\
y_2(2x_1) & \leq y_2(10) \\
y_3(x_1 + x_2) & \leq y_3(7) \\
y_4(x_1) & \leq y_4(5) \\
(y_1 + 2y_2 + y_3 + y_4)x_1 + (3y_1 - 4y_2 + y_3)x_2 & \leq \ldots
\end{align*}
\]

2. $5y_1 + 10y_2 + 7y_3 + 5y_4$ is an upper bound,
Generalizing ...

1. Multiply first equation by $y_1$, second by $y_2$, third by $y_3$ and fourth by $y_4$ ($y_1, y_2, y_3, y_4 \geq 0$) and add

$$y_1(x_1 + 3x_2) \leq y_1(5)$$
$$+ y_2(2x_1 - 4x_2) \leq y_2(10)$$
$$+ y_3(x_1 + x_2) \leq y_3(7)$$
$$+ y_4(x_1) \leq y_4(5)$$

$$(y_1 + 2y_2 + y_3 + y_4)x_1 + (3y_1 - 4y_2 + y_3)x_2 \leq \ldots$$

2. $5y_1 + 10y_2 + 7y_3 + 5y_4$ is an upper bound, provided coefficients of $x_i$ are same as in the objective function $(4x_1 + 2x_2)$,

$$y_1 + 2y_2 + y_3 + y_4 = 4 \quad 3y_1 - 4y_2 + y_3 = 2$$
Generalizing...

1. Multiply first equation by $y_1$, second by $y_2$, third by $y_3$ and fourth by $y_4$ ($y_1, y_2, y_3, y_4 \geq 0$) and add

\[
\begin{align*}
   y_1( & x_1 + 3x_2 ) \leq y_1(5) \\
   + y_2( & 2x_1 - 4x_2 ) \leq y_2(10) \\
   + y_3( & x_1 + x_2 ) \leq y_3(7) \\
   + y_4( & x_1 ) \leq y_4(5) \\
\end{align*}
\]
\[
(y_1 + 2y_2 + y_3 + y_4)x_1 + (3y_1 - 4y_2 + y_3)x_2 \leq \ldots
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2. $5y_1 + 10y_2 + 7y_3 + 5y_4$ is an upper bound, provided coefficients of $x_i$ are same as in the objective function $(4x_1 + 2x_2)$,

\[
y_1 + 2y_2 + y_3 + y_4 = 4 \quad 3y_1 - 4y_2 + y_3 = 2
\]

3. Subject to these constrains, the best upper bound is

\[
\min : 5y_1 + 10y_2 + 7y_3 + 5y_4!
\]
Dual LP: Example

Thus, the optimum value of program

\[
\begin{align*}
\text{maximize} & \quad 4x_1 + 2x_2 \\
\text{subject to} & \quad x_1 + 3x_2 \leq 5 \\
& \quad 2x_1 - 4x_2 \leq 10 \\
& \quad x_1 + x_2 \leq 7 \\
& \quad x_1 \leq 5
\end{align*}
\]

is upper bounded by the optimal value of the program

\[
\begin{align*}
\text{minimize} & \quad 5y_1 + 10y_2 + 7y_3 + 5y_4 \\
\text{subject to} & \quad y_1 + 2y_2 + y_3 + y_4 = 4 \\
& \quad 3y_1 - 4y_2 + y_3 = 2 \\
& \quad y_1, y_2 \geq 0
\end{align*}
\]
Dual Linear Program

Given a linear program \( \Pi \) in canonical form

\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{d} c_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad i = 1, 2, \ldots, n
\end{align*}
\]

the dual \( \text{Dual}(\Pi) \) is given by

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{n} b_i y_i \\
\text{subject to} & \quad \sum_{i=1}^{n} y_i a_{ij} = c_j \quad j = 1, 2, \ldots, d \\
y_i \geq 0 & \quad i = 1, 2, \ldots, n
\end{align*}
\]
Given a linear program \( \Pi \) in canonical form

maximize \( \sum_{j=1}^{d} c_j x_j \)
subject to \( \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad i = 1, 2, \ldots n \)

the dual \( \text{Dual}(\Pi) \) is given by

minimize \( \sum_{i=1}^{n} b_i y_i \)
subject to \( \sum_{i=1}^{n} y_i a_{ij} = c_j \quad j = 1, 2, \ldots d \)
\( y_i \geq 0 \quad i = 1, 2, \ldots n \)

**Proposition**

\( \text{Dual(Dual}(\Pi)) \) is equivalent to \( \Pi \)
Given a $A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^d$, linear program $\Pi$

$$\text{maximize } c \cdot x$$
$$\text{subject to } Ax \leq b$$

the dual $\text{Dual}(\Pi)$ is given by

$$\text{minimize } y \cdot b$$
$$\text{subject to } yA = c$$
$$y \geq 0$$

Proposition

$\text{Dual(Dual(\Pi))}$ is equivalent to $\Pi$
Theorem (Weak Duality)

If $x$ is a feasible solution to $\Pi$ and $y$ is a feasible solution to $\text{Dual}(\Pi)$ then $c \cdot x \leq y \cdot b$. 

Theorem (Strong Duality)

If $x^*$ is an optimal solution to $\Pi$ and $y^*$ is an optimal solution to $\text{Dual}(\Pi)$ then $c \cdot x^* = y^* \cdot b$. 

Many applications! Maxflow-Mincut theorem can be deduced from duality.
Duality Theorem

Theorem (Weak Duality)

If \( x \) is a feasible solution to \( \Pi \) and \( y \) is a feasible solution to \( \text{Dual}(\Pi) \) then \( c \cdot x \leq y \cdot b \).

Theorem (Strong Duality)

If \( x^* \) is an optimal solution to \( \Pi \) and \( y^* \) is an optimal solution to \( \text{Dual}(\Pi) \) then \( c \cdot x^* = y^* \cdot b \).

Many applications! Maxflow-Mincut theorem can be deduced from duality.
Weak Duality

**Theorem (Weak Duality)**

If \( x \) is a feasible solution to \( \Pi \) and \( y \) is a feasible solution to \( \text{Dual}(\Pi) \) then \( c \cdot x \leq y \cdot b \).

We already saw the proof by the way we derived it but we will do it again formally.

**Proof.**

Since \( y' \) is feasible in \( \text{Dual}(\Pi) \): \( y'A = c \)
Weak Duality

Theorem (Weak Duality)

If $x$ is a feasible solution to $\Pi$ and $y$ is a feasible solution to $\text{Dual}(\Pi)$ then $c \cdot x \leq y \cdot b$.

We already saw the proof by the way we derived it but we will do it again formally.

Proof.

Since $y'$ is feasible in $\text{Dual}(\Pi)$: $y' A = c$

Therefore $c \cdot x' = y' A x'$
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Proof.

Since \( y' \) is feasible in \( \text{Dual}(\Pi) \): \( y' A = c \)

Therefore \( c \cdot x' = y' A x' \)

Since \( x' \) is feasible in \( \Pi \), \( A x' \leq b \) and hence,

\[
c \cdot x' = y' A x' \leq y' \cdot b
\]
Strong Duality and Complementary Slackness

\begin{align*}
\text{maximize :} & \quad c \cdot x \\
\text{subject to} & \quad Ax \leq b
\end{align*}
\begin{align*}
\text{Dual} \\
\begin{align*}
\text{minimize :} & \quad y \cdot b \\
\text{subject to} & \quad yA = c \\
& \quad y \geq 0
\end{align*}
\end{align*}

**Definition (Complementary Slackness)**

\(x\) feasible in \(\Pi\) and \(y\) feasible in \(\text{Dual}(\Pi)\), s.t., 
\[
\forall i = 1..n, \quad y_i > 0 \implies (Ax)_i = b_i
\]
Strong Duality and Complementary Slackness

\[
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**Geoemetric Interpretation:** $c$ is in the cone of the normal vectors of the tight hyperplanes at $x$. 
Strong Duality and Complementary Slackness

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$x$ feasible in $\Pi$ and $y$ feasible in $\text{Dual}(\Pi)$, s.t.,

\[ \forall i = 1..n, \quad y_i > 0 \Rightarrow (Ax)_i = b_i \]

**Theorem**

$(x^*, y^*)$ satisfies complementary Slackness if and only if strong duality holds, i.e., $c \cdot x^* = y^* \cdot b$.

**Proof.**

\[
\begin{align*}
    c \cdot x^* &= (y^* A) \cdot x^* \\
               &= y^* \cdot (Ax^*)
\end{align*}
\]

$(\Rightarrow)$
**Strong Duality and Complementary Slackness**

**Definition (Complementary Slackness)**

\[ x \text{ feasible in } \Pi \text{ and } y \text{ feasible in } \text{Dual}(\Pi), \text{ s.t.,} \]

\[ \forall i = 1..n, \quad y_i > 0 \implies (Ax)_i = b_i \]

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**Proof.**

\[
\begin{align*}
c \cdot x^* &= (y^* A) \cdot x^* \\
&= y^* \cdot (Ax^*) \\
&= \sum_{i=1}^{n} y_i^* (Ax^*)_i \\
&= \sum_{i:y_i > 0} y_i^* (Ax^*)_i
\end{align*}
\]
**Definition (Complementary Slackness)**

\(x\) feasible in \(\Pi\) and \(y\) feasible in \(\text{Dual}(\Pi)\), s.t.,
\[
\forall i = 1..n, \quad y_i > 0 \Rightarrow (Ax)_i = b_i
\]

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\[
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\[
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= y^* \cdot (Ax^*) \\
= \sum_{i=1}^{n} y_i^* (Ax^*)_i \\
= \sum_{i: y_i > 0} y_i^* (Ax^*)_i \\
= \sum_i y_i^* b_i = y^* \cdot b
\]
Strong Duality and Complementary Slackness

Definition (Complementary Slackness)

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**Proof.**

(\( \Leftarrow \)) **Exercise**
Duality for another canonical form

maximize $4x_1 + x_2 + 3x_3$
subject to $x_1 + 4x_2 \leq 2$
$2x_1 - x_2 + x_3 \leq 4$
$x_1, x_2, x_3 \geq 0$
Duality for another canonical form

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Choose non-negative \( y_1, y_2 \) and multiply inequalities

maximize \( 4x_1 + x_2 + 3x_3 \)
subject to \( y_1(x_1 + 4x_2) \leq 2y_1 \)
\( y_2(2x_1 - x_2 + x_3) \leq 4y_2 \)
\( x_1, x_2, x_3 \geq 0 \)
Duality for another canonical form

Choose non-negative $y_1, y_2$ and multiply inequalities

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$y_2(2x_1 - x_2 + x_3) \leq 4y_2$
$x_1, x_2, x_3 \geq 0$

Adding the inequalities we get an inequality below that is valid for any feasible $x$ and any non-negative $y$:

$$(y_1 + 2y_2)x_1 + (4y_1 - y_2)x_2 + y_2x_3 \leq 2y_1 + 4y_2$$
Duality for another canonical form

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$$(y_1 + 2y_2)x_1 + (4y_1 - y_2)x_2 + y_2x_3 \leq 2y_1 + 4y_2$$

Suppose we choose $y_1, y_2$ such that $y_1 + 2y_2 \geq 4$ and $4y_2 - y_2 \geq 1$ and $y_2 \geq 3$

Then, since $x_1, x_2, x_3 \geq 0$, we have $4x_1 + x_2 + 3x_3 \leq 2y_1 + 4y_2$
Duality for another canonical form

maximize \[ 4x_1 + x_2 + 3x_3 \]
subject to \[ x_1 + 4x_2 \leq 2 \]
\[ 2x_1 - x_2 + x_3 \leq 4 \]
\[ x_1, x_2, x_3 \geq 0 \]

is upper bounded by

minimize \[ 2y_1 + 4y_2 \]
subject to \[ y_1 + 2y_2 \geq 4 \]
\[ 4y_1 - y_2 \geq 1 \]
\[ y_2 \geq 3 \]
\[ y_1, y_2 \geq 0 \]
Duality for another canonical form

Compactly, for the primal LP $\Pi$

$$\begin{align*}
\text{max} \quad & c \cdot x \\
\text{subject to} \quad & Ax \leq b, \ x \geq 0
\end{align*}$$

the dual LP is $\text{Dual}(\Pi)$

$$\begin{align*}
\text{min} \quad & y \cdot b \\
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Compactly, for the primal LP $\Pi$

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**Definition (Complementary Slackness)**

$x$ feasible in $\Pi$ and $y$ feasible in $\text{Dual}(\Pi)$, s.t.,

$\forall i = 1, \ldots, n, \ y_i > 0 \Rightarrow (Ax)_i = b_i$

$\forall j = 1, \ldots, d, \ x_j > 0 \Rightarrow (yA)_j = c_j$
In General...
from Jeff’s notes

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
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</thead>
<tbody>
<tr>
<td>( \text{max } c \cdot x )</td>
<td>( \text{min } y \cdot b )</td>
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<tr>
<td>( \sum_j a_{ij} x_j \leq b_i )</td>
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<td>( x_j \geq 0 )</td>
<td>( \sum_i y_i a_{ij} \geq c_j )</td>
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</table>

**Figure H.4.** Constructing the dual of an arbitrary linear program.
Some Useful Duality Properties

Assume primal LP is a maximization LP.

- For a given LP, Dual is another LP. The variables in the dual correspond to “tight” primal constraints and vice-versa.
Some Useful Duality Properties

Assume primal LP is a maximization LP.

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- If primal is unbounded (objective achieves infinity) then dual LP is infeasible. Why? If dual LP had a feasible solution it would upper bound the primal LP which is not possible.
- If primal is infeasible then dual LP is unbounded.
- Primal and dual optimum solutions satisfy complementary slackness conditions (discussed soon).
Part II

Examples of Duality
Max matching in bipartite graph as LP

Input: $G = (V = L \cup R, E)$

\[
\begin{align*}
\text{max} \quad & \sum_{uv \in E} x_{uv} \\
\text{s.t.} \quad & \sum_{uv \in E} x_{uv} \leq 1 \quad \forall v \in V. \\
\quad & x_{uv} \geq 0 \quad \forall uv \in E
\end{align*}
\]

When one writes combinatorial problems as LPs one is writing a single formulation in an abstract way that applies to all instances. In the above, for each fixed graph $G$ one gets a fixed LP and hence the above is sometimes called a “formulation”.
Max matching in bipartite graph as LP

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\[
\begin{align*}
\text{max} & \quad \sum_{uv \in E} x_{uv} \\
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& \quad x_{uv} \geq 0 \quad \forall uv \in E
\end{align*}
\]

Dual LP has one variable \( y_v \) for each vertex \( v \in V \).

\[
\begin{align*}
\text{min} & \quad \sum_{v \in V} y_v \\
\text{s.t.} & \quad y_u + y_v \geq 1 \quad \forall uv \in E \\
& \quad y_v \geq 0 \quad \forall v \in V
\end{align*}
\]
Network flow

$s$-$t$ flow in directed graph $G = (V, E)$ with capacities $c$. Assume for simplicity that no incoming edges into $s$.

$$\max \sum_{(s, v) \in E} x(s, v)$$

$$\sum_{(u, v) \in E} x(u, v) - \sum_{(v, w) \in E} x(v, w) = 0 \quad \forall v \in V \setminus \{s, t\}$$

$$x(u, v) \leq c(u, v) \quad \forall (u, v) \in E$$

$$x(u, v) \geq 0 \quad \forall (u, v) \in E.$$
Dual of Network Flow