CS 473: Algorithms

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High Probability Analysis & Universal Hashing

Lecture 09
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Most slides are courtesy Prof. Chekuri
Randomized **QuickSort** w.h.p. (any questions?)

What is the probability that the algorithm will terminate in $O(n \log n)$ time?

**Balls & Bins**
- Expected bin size.
- Expected max bin size $\rightarrow$ max size w.h.p.
- Analogy to hashing

**Hashing**
Part I

Randomized QuickSort (Contd.)
Randomized QuickSort: Recall

**Input:** Array $A$ of $n$ distinct numbers. **Output:** Numbers in sorted order.

### Randomized QuickSort

1. Pick a pivot element *uniformly at random* from $A$.
2. Split array into 2 subarrays: those smaller than pivot (L), and those larger than pivot (R).
3. Recursively sort the subarrays, and concatenate them.
Randomized **QuickSort**: Recall

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**Note**: On every input randomized QuickSort takes $O(n \log n)$ time in expectation. On every input it may take $\Omega(n^2)$ time with some small probability.
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**Note:** On *every* input randomized QuickSort takes $O(n \log n)$ time in expectation. On *every* input it may take $\Omega(n^2)$ time with some small probability.

Question: With what probability it takes $O(n \log n)$ time?
Informal Statement

Random variable $Q(A) = \#$ comparisons done by the algorithm.

We will show that $\Pr[Q(A) \leq 32n \ln n] \geq 1 - \frac{1}{n^3}$. 
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If $n = 100$ then this gives $\Pr[Q(A) \leq 32n \ln n] \geq 0.99999$. 
Informal Statement
We will show that $\Pr[Q(A) \leq 32n \ln n] \geq 1 - \frac{1}{n^3}$.

Outline of the proof
- If depth of recursion is $k$ then $Q(A) \leq kn$.
- Prove that depth of recursion $\leq 32 \ln n$ with high probability (w.h.p.). This will imply the result.
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  1. Focus on a single element. Prove that it “participates” in $> 32 \ln n$ levels with probability (w.p.) at most $\frac{1}{n^4}$.
  2. By union bound, any of the $n$ elements participates in $> 32 \ln n$ levels w.p. at most $\frac{1}{n^4}$. 
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Randomized **QuickSort**: High Probability Analysis

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   1. Focus on a single element. Prove that it “participates” in $> 32 \ln n$ levels with probability (w.p.) at most $\frac{1}{n^4}$.
   2. By union bound, any of the $n$ elements participates in $> 32 \ln n$ levels w.p. at most $\frac{1}{n^3}$.
3. Therefore, all elements participate in $\leq 32 \ln n$ w.p. $(1 - \frac{1}{n^3})$. 
**Informal Statement**

An element participates in $> 32 \ln n$ w.p. $\leq 1/n^4$.

**Intuition**

1. When we pick a pivot from an array of size $n$ uniformly at random, what is the probability that its rank is between $n/4$ and $3n/4$?
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An element participates in $\geq 32 \ln n$ w.p. $\leq 1/n^4$.

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1. When we pick a pivot from an array of size $n$ uniformly at random, what is the probability that its rank is between $n/4$ and $3n/4$? 1/2.
2. If we pick such a pivot then the size of L and R is at most?
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Randomized **QuickSort**: High Probability Analysis

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3. If an array is reduced to at least its $3/4$th size every time, then after how many rounds only one element remains? $\leq 4 \ln n$.

4. If $32 \ln n$ splits, then $\mathbb{E}[\text{Balanced-split}] = 16 \ln n$. Out of these there are $< 4 \ln n$ balanced split w.p. $\leq 1/n^4$. 
Randomized **QuickSort**: High Probability Analysis

- If $k$ levels of recursion then $kn$ comparisons.
Randomized QuickSort: High Probability Analysis

- If $k$ levels of recursion then $kn$ comparisons.
- Fix an element $s \in A$. We will track it at each level.
- Let $S_i$ be the partition containing $s$ at $i^{th}$ level.
- $S_1 = A$ and $S_k = \{s\}$. 
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We call $s$ lucky in $i^{th}$ iteration, if balanced split:
\[ |S_{i+1}| \leq (3/4)|S_i| \text{ and } |S_i \setminus S_{i+1}| \leq (3/4)|S_i|. \]
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- If $\rho = \#$ lucky rounds in first $k$ rounds, then $|S_k| \leq (3/4)^\rho n$. 
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$$|S_{i+1}| \leq \left(\frac{3}{4}\right)|S_i| \text{ and } |S_i \setminus S_{i+1}| \leq \left(\frac{3}{4}\right)|S_i|.$$ 

- If $\rho = \#$lucky rounds in first $k$ rounds, then
  $$|S_k| \leq \left(\frac{3}{4}\right)^\rho n.$$ 

- For $|S_k| = 1$, $\rho = 4 \ln n \geq \log_{4/3} n$ suffices.
How may rounds before $4 \ln n$ lucky rounds?

- $X_i = 1$ if $s$ is lucky in $i^{th}$ iteration.
How many rounds before $4 \ln n$ lucky rounds?

- $X_i = 1$ if $s$ is lucky in $i^{th}$ iteration.
- Observation: $X_1, \ldots, X_k$ are independent variables.
- $\Pr[X_i = 1] = \frac{1}{2}$ Why?

Clearly, $\rho = \sum_{i=1}^{k} X_i$. Let $\mu = E[\rho] = k/2$.

Set $k = 32 \ln n$ and $\delta = \frac{3}{4}$.

$(1 - \delta) = \frac{1}{4}$.

Probability of $\leq 4 \ln n$ lucky rounds out of $32 \ln n$ rounds is,

$$\Pr[\rho \leq 4 \ln n] = \Pr[\rho \leq k/8] = \Pr[\rho \leq (1 - \delta)\mu] \quad (\text{Chernoff})$$

$$\leq 2e^{-\delta^2\mu^2} = 2e^{-\frac{9}{64}k} = 2e^{-4.5 \ln n} \leq \frac{1}{n^4}.$$
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Set \( k = 32 \ln n \) and \( \delta = \frac{3}{4} \). \( (1 - \delta) = \frac{1}{4} \).

Probability of \( \leq 4 \ln n \) lucky rounds out of \( 32 \ln n \) rounds is,

\[
\Pr[\rho \leq 4 \ln n] = \Pr[\rho \leq k/8] = \Pr[\rho \leq (1 - \delta)\mu] \leq 2e^{-\frac{\delta^2}{2}\mu} = 2e^{-\frac{9k}{64}} = 2e^{-4.5 \ln n} \leq \frac{1}{n^4}
\]
Randomized **QuickSort** w.h.p. Analysis

- **n** input elements. Probability that depth of recursion in **QuickSort** $> 32 \ln n$ is at most $\frac{1}{n^4} \times n = \frac{1}{n^3}$. 
Randomized **QuickSort** w.h.p. Analysis

- $n$ input elements. Probability that depth of recursion in **QuickSort** $\geq 32 \ln n$ is at most $\frac{1}{n^4} \times n = \frac{1}{n^3}$.

**Theorem**

*With high probability (i.e., $1 - \frac{1}{n^3}$) the depth of the recursion of **QuickSort** is $\leq 32 \ln n$. Due to $n$ comparisons in each level, with high probability, the running time of **QuickSort** is $O(n \ln n)$.***
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- $n$ input elements. Probability that depth of recursion in **QuickSort** $\geq 32 \ln n$ is at most $\frac{1}{n^4} \times n = \frac{1}{n^3}$.

**Theorem**

*With high probability (i.e., $1 - \frac{1}{n^3}$) the depth of the recursion of **QuickSort** is $\leq 32 \ln n$. Due to $n$ comparisons in each level, with high probability, the running time of **QuickSort** is $O(n \ln n)$.***

**Q:** How to increase the probability?
Part II

Balls and Bins
Expected Bin Size

Problem
If \( n \) balls are thrown independently and uniformly into \( n \) bins, how many balls land in a bin in expectation (expected size of a bin)?
Expected Bin Size

Problem
If $n$ balls are thrown independently and uniformly into $n$ bins, how many balls lend in a bin in expectation (expected size of a bin)?

Solution
- Fix a bin, say $j$. 

\[ E[X_{ij}] = \Pr[X_{ij} = 1] = \frac{1}{n}. \]

\[ Y_j = \text{# balls in } j \text{th bin} = \sum_{i=1}^{n} X_{ij}. \]

\[ E[Y_j] = \sum_{i=1}^{n} E[X_{ij}] = n \cdot \frac{1}{n} = 1. \]
Problem
If \( n \) balls are thrown independently and uniformly into \( n \) bins, how many balls lend in a bin in expectation (expected size of a bin)?

Solution
- Fix a bin, say \( j \).
- Random variable \( X_{ij} \) is 1 if \( i \)th balls falls in \( j \)th bin, otherwise 0.
Expected Bin Size

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If $n$ balls are thrown independently and uniformly into $n$ bins, how many balls lend in a bin in expectation (expected size of a bin)?

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- \( \mathbb{E}[X_{ij}] = \mathbb{P}[X_{ij} = 1] = 1/n \).
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- \( E[X_{ij}] = \Pr[X_{ij} = 1] = 1/n \).
- R.V. \( Y_j = \# \) balls in \( j \)th bin = \( \sum_{i=1}^{n} X_{ij} \).
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- $\mathbb{E}[Y_j] = \sum_{i=1}^{n} \mathbb{E}[X_{ij}] = n \cdot 1/n = 1$. 
Expected Max Bin Size

Problem

If $n$ balls are thrown independently and uniformly into $n$ bins, what is the expected “maximum” bin size?

Possible Solution

R.V. $Z = \max_{1 \leq j \leq n} Y_j$. 

$E[Z] = \sum_{k=1}^{n} Pr[Z = k] k$.

How to compute $Pr[Z = k]$, i.e., count configurations where no bin has more than $k$ balls and at least one has $k$ balls.

Too many to count!!
Expected Max Bin Size

Problem
If \( n \) balls are thrown independently and uniformly into \( n \) bins, what is the expected “maximum” bin size?

\[
E\left[\max_{j=1}^{n} Y_j\right]
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Expected Max Bin Size

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$$E\left[ \max_{j=1}^{n} Y_j \right]?$$

Possible Solution
- R.V. $Z = \max_{j=1}^{n} Y_j$. $E[Z] = \sum_{k=1}^{n} \Pr[Z = k] k$. 
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- Too many to count!!
Expected Max Bin Size (Contd.)

Problem

What is the expected maximum bin size?

R.V. \( Z = \max_{j=1}^{n} Y_j \). Show \( E[Z] \leq O \left( \frac{\ln n}{\ln \ln n} \right) \)?

Possible Solution

- If \( \Pr[Z > \frac{8 \ln n}{\ln \ln n}] \leq 1/n^2 \), then: define \( A = \frac{8 \ln n}{\ln \ln n} \).
**Problem**

What is the expected maximum bin size?

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\[
E[Z] \leq \sum_{k=1}^{A} \Pr[Z = k] A + \sum_{k=A+1}^{n} \Pr[Z = k] n \\
\leq A \cdot \Pr[Z \leq A] + n \cdot \Pr[Z > A] \\
\leq A \cdot (1) + n \cdot (1/n^2) = O(A) = O \left( \frac{\ln n}{\ln \ln n} \right)
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Expected Max Bin Size (Contd.)

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What is the expected maximum bin size?

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\begin{align*}
\mathbb{E}[Z] & \leq \sum_{k=1}^{A} \Pr[Z = k] A + \sum_{k=A+1}^{n} \Pr[Z = k] n \\
& \leq A \cdot \Pr[Z \leq A] + n \cdot \Pr[Z > A] \\
& \leq A \cdot (1) + n \cdot (1/n^2) = O(A) = O \left( \frac{\ln n}{\ln \ln n} \right)
\end{align*}
\]

Bound \( \Pr[Z > \frac{8 \ln n}{\ln \ln n}] \).
Bound $\Pr[Z > \frac{8 \ln n}{\ln \ln n}]$ using Chernoff inequality.

Chernoff Inequality. We Saw

$X_1, \ldots, X_k$ independent binary R.V., and $X = \sum_{i=1}^{k} X_i$, $\mu = \mathbb{E}[X]$, then for $0 < \delta < 1$

$$
\Pr[X \geq (1 + \delta)\mu] \leq e^{-\delta^2 \mu / 3} \quad \& \quad \Pr[X \leq (1 - \delta)\mu] \leq e^{-\delta^2 \mu / 2}
$$
Expected Max Bin Size (Contd.)

Bound \( \Pr[Z > \frac{8 \ln n}{\ln \ln n}] \) using Chernoff inequality.

Chernoff Ineq. We Saw

\( X_1, \ldots, X_k \) independent binary R.V., and \( X = \sum_{i=1}^{k} X_i \), \( \mu = \mathbb{E}[X] \), then for \( 0 < \delta < 1 \)

\[ \Pr[X \geq (1 + \delta)\mu] \leq e^{-\delta^2 \mu / 3} \quad \& \quad \Pr[X \leq (1 - \delta)\mu] \leq e^{-\delta^2 \mu / 2} \]

Stronger Versions

- For \( \delta > 0 \), \( \Pr[X > (1 + \delta)\mu] < \left( \frac{e^\delta}{(1+\delta)^{(1+\delta)}} \right)^{\mu} \).
- For \( 0 < \delta < 1 \) \( \Pr[X < (1 - \delta)\mu] < \left( \frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}} \right)^{\mu} \).
Problem

What is the expected maximum bin size? Let $Z = \max_{j=1}^{n} Y_j$.

Show $\mathbb{E}[Z] \leq O\left(\frac{\ln n}{\ln \ln n}\right)$. → Show $\Pr[Z > \frac{8 \ln n}{\ln \ln n}] \leq \frac{1}{n^2}$.
Problem
What is the expected maximum bin size? Let $Z = \max_{j=1}^n Y_j$.

Show $E[Z] \leq O\left(\frac{\ln n}{\ln \ln n}\right)$. → Show $\Pr[Z > \frac{8 \ln n}{\ln \ln n}] \leq 1/n^2$.

Solution
Recall: $Y_j = \#$ balls in bin $j$, $E[Y_j] = 1$, and $A = \frac{8 \ln n}{\ln \ln n}$

$$\Pr[Y_j > A] = \Pr[Y_j \geq A E[Y]] < \left(\frac{e^{A-1}}{A^A}\right) < \left(\frac{n^6/\ln \ln n}{A^A}\right)$$
Expected Max Bin Size (Contd.)

Problem

What is the expected maximum bin size? Let \( Z = \max_{j=1}^{n} Y_j \).

Show \( E[Z] \leq O\left(\frac{\ln n}{\ln \ln n}\right) \).

Show \( \Pr[Z > \frac{8 \ln n}{\ln \ln n}] \leq 1/n^2 \).

Solution

Recall: \( Y_j = \# \text{ balls in bin } j \), \( E[Y_j] = 1 \), and \( A = \frac{8 \ln n}{\ln \ln n} \)

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\Pr[Y_j > A] = \Pr[Y_j \geq A E[Y]] < \left(\frac{e^{A-1}}{A^A}\right) < \left(\frac{n^{6/\ln \ln n}}{A^A}\right)
\]

\[
A^A = \left(\frac{8 \ln n}{\ln \ln n}\right)^{\frac{8 \ln n}{\ln \ln n}} \geq (\sqrt{\ln n})^{\frac{8 \ln n}{\ln \ln n}} = (\ln n)^{\frac{4 \ln n}{\ln \ln n}} = e^{4 \lg n} = n^4
\]
Problem

What is the expected maximum bin size? Let \( Z = \max_{j=1}^{n} Y_j \).
Show \( E[Z] \leq O\left(\frac{\ln n}{\ln \ln n}\right) \). → Show \( \Pr[Z > \frac{8 \ln n}{\ln \ln n}] \leq 1/n^2 \).

Solution

Recall: \( Y_j = \# \) balls in bin \( j \), \( E[Y_j] = 1 \), and \( A = \frac{8 \ln n}{\ln \ln n} \)

\[
\Pr[Y_j > A] = \Pr[Y_j \geq A E[Y]] < \left(\frac{e^{A-1}}{A^A}\right) < \left(\frac{n^{6/\ln \ln n}}{A^A}\right)
\]

\[
A^A = \left(\frac{8 \ln n}{\ln \ln n}\right)^{\frac{8 \ln n}{\ln \ln n}} \geq (\sqrt{\ln n})^{\frac{8 \ln n}{\ln \ln n}} = (\ln n)^{\frac{4 \ln n}{\ln \ln n}} = e^{4 \lg n} = n^4
\]

\[
\Pr[Y_j > \frac{8 \ln n}{\ln \ln n}] < 1/n^3
\]
Expected Max Bin Size (Contd.)

Problem
What is the expected maximum bin size? Let $Z = \max_{j=1}^{n} Y_j$.
Show $E[Z] \leq O(\frac{\ln n}{\ln \ln n})$. → Show $\Pr[Z > \frac{8 \ln n}{\ln \ln n}] \leq 1/n^2$.

Solution
$\Pr[Y_j > 8 \ln n / \ln \ln n] \leq 1/n^3$ (Using Chernoff)
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What is the expected maximum bin size? Let $Z = \max_{j=1}^{n} Y_j$.
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Solution


$\Pr[Y_j > 8 \ln n / \ln \ln n] \leq 1/n^3$    (Using Chernoff)

(Union bound)

$\Pr\left[Z > \frac{8 \ln n}{\ln \ln n}\right] \leq \sum_{j=1}^{n} \Pr[Y_j > \frac{8 \ln n}{\ln \ln n}] \leq n \cdot 1/n^3 = 1/n^2$.
Expected Max Bin Size (Contd.)

**Problem**

What is the expected maximum bin size? Let $Z = \max_{j=1}^{n} Y_j$.

Show $\mathbf{E}[Z] \leq \mathcal{O}\left(\frac{\ln n}{\ln \ln n}\right)$. \(\rightarrow\) Show $\mathbf{Pr}\left[Z > \frac{8 \ln n}{\ln \ln n}\right] \leq 1/n^2$.

**Solution**

- Recall: $Y_j = \# \text{ balls in bin } j$. $\mathbf{E}[Y_j] = 1$.

  $\mathbf{Pr}[Y_j > \frac{8 \ln n}{\ln \ln n}] \leq 1/n^3$ \hspace{1cm} (Using Chernoff)

- (Union bound)

  $\mathbf{Pr}\left[Z > \frac{8 \ln n}{\ln \ln n}\right] \leq \sum_{j=1}^{n} \mathbf{Pr}\left[Y_j > \frac{8 \ln n}{\ln \ln n}\right] \leq n \cdot 1/n^3 = 1/n^2$.

- Max bin size is at most $\mathcal{O}\left(\frac{\ln n}{\ln \ln n}\right)$ with probability $1 - 1/n^2$. 

Ω(\ln n / \ln \ln n) is a lower bound as well!
Problem

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- Recall: $Y_j = \#$ balls in bin $j$. $\mathbb{E}[Y_j] = 1$.
  
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  Max bin size is at most $O\left(\frac{\ln n}{\ln \ln n}\right)$ with probability $1 - 1/n^2$.

$\Omega\left(\frac{\ln n}{\ln \ln n}\right)$ is a lower bound as well!
Hashing

Storing elements in a table such that look up is $O(1)$-time.
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Throwing numbered balls

Imagine that $n$ balls have numbers coming from a universe $U$. $|U| \gg n$. 
Hashing

Storing elements in a table such that look up is $O(1)$-time.

Throwing numbered balls

Imagine that $n$ balls have numbers coming from a universe $\mathcal{U}$. 
$|\mathcal{U}| \gg n$.

Hashing: throw balls (elements) randomly into $n$ bins such that bin sizes are small.
Hashing

Storing elements in a table such that look up is $O(1)$-time.

Throwing numbered balls

Imagine that $n$ balls have numbers coming from a universe $\mathcal{U}$. $|\mathcal{U}| \gg n$.

Hashing: throw balls (elements) randomly into $n$ bins such that bin sizes are small and also lookup is easy!
Part III

Hash Tables
**Dictionary Data Structure**

1. $\mathcal{U}$: universe of keys with total order: numbers, strings, etc.
2. Data structure to store a subset $S \subseteq \mathcal{U}$
3. **Operations:**
   1. **Search/lookup:** given $x \in \mathcal{U}$ is $x \in S$?
   2. **Insert:** given $x \not\in S$ add $x$ to $S$.
   3. **Delete:** given $x \in S$ delete $x$ from $S$
4. **Static** structure: $S$ given in advance or changes very infrequently, main operations are lookups.
5. **Dynamic** structure: $S$ changes rapidly so inserts and deletes as important as lookups.
Common solutions:

1. **Static:**
   - Store $S$ as a *sorted* array
   - **Lookup**: Binary search in $O(\log |S|)$ time (comparisons)

2. **Dynamic:**
   - Store $S$ in a *balanced* binary search tree
   - Lookup, Insert, Delete in $O(\log |S|)$ time (comparisons)
Question: “Should Tables be Sorted?”
(also title of famous paper by Turing award winner Andy Yao)
Dictionary Data Structures

**Question:** “Should Tables be Sorted?”
(Also title of famous paper by Turing award winner Andy Yao)

Hashing is a widely used & powerful technique for dictionaries.

**Motivation:**

1. Universe $\mathcal{U}$ may not be (naturally) totally ordered.
2. Keys correspond to large objects (images, graphs etc) for which comparisons are very expensive.
3. Want to improve “average” performance of lookups to $O(1)$ even at cost of extra space or errors with small probability: many applications for fast lookups in networking, security, etc.
Hashing and Hash Tables

Hash Table data structure:

1. A (hash) table/array $T$ of size $m$ (the table size).
2. A hash function $h : U \rightarrow \{0, \ldots, m - 1\}$.
3. Item $x \in U$ hashes to slot $h(x)$ in $T$. 

Given $S \subseteq U$. How do we store $S$ and how do we do lookups?

Ideal situation:

1. Each element $x \in S$ hashes to a distinct slot in $T$.
2. Lookup: Given $y \in U$ check if $T[h(y)] = y$.

$O(1)$ time!

Collisions unavoidable if $|T| < |U|$. Several techniques to handle them.
Hashing and Hash Tables

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2. **Lookup**: Given $y \in \mathcal{U}$ check if $T[h(y)] = y$. $O(1)$ time!

Collisions unavoidable if $|T| < |\mathcal{U}|$. Several techniques to handle them.
Handling Collisions: Chaining

Collision: \( h(x) = h(y) \) for some \( x \neq y \).

**Chaining** to handle collisions:

1. For each slot \( i \) store all items hashed to slot \( i \) in a linked list. \( T[i] \) points to the linked list.
2. **Lookup**: to find if \( y \in U \) is in \( T \), check the linked list at \( T[h(y)] \). Time proportion to size of linked list.

This is also known as **Open hashing**.
Handling Collisions

Several other techniques:

1. Cuckoo hashing. Every value has two possible locations. When inserting, insert in one of the locations, otherwise, kick stored value to its other location. Repeat till stable. if no stability then rebuild table.

2. . . .

3. Others.
Does hashing give $O(1)$ time per operation for dictionaries?
Understanding Hashing

Does hashing give $O(1)$ time per operation for dictionaries?

Questions:

1. Complexity of evaluating $h$ on a given element?
2. Relative sizes of the universe $U$ and the set to be stored $S$.
3. Size of table relative to size of $S$.
4. Worst-case vs average-case vs randomized (expected) time?
5. How do we choose $h$?
Complexity of evaluating $h$ on a given element? Should be small.

2. Relative sizes of the universe $U$ and the set to be stored $S$: typically $|U| \gg |S|$.

3. Size of table relative to size of $S$. The load factor of $T$ is the ratio $n/m$ where $n = |S|$ and $m = |T|$. Typically $n/m$ is a small constant smaller than 1. Also known as the fill factor.
Understanding Hashing

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Main and interrelated questions:
1. Worst-case vs average-case vs randomized (expected) time?
2. How do we choose \( h \)?
Single hash function

1. $\mathcal{U}$: universe (very large).
2. Assume $N = |\mathcal{U}| \gg m$ where $m$ is size of table $T$. In particular assume $N \geq m^2$ (very conservative).
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2. Assume $N = |\mathcal{U}| \gg m$ where $m$ is size of table $T$. In particular assume $N \geq m^2$ (very conservative).

3. Fix hash function $h : \mathcal{U} \rightarrow \{0, \ldots, m - 1\}$. 

Lesson: For every hash function there is a very bad set. Bad set. Bad.
**Single hash function**

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5. Implies that there is a set $S \subseteq \mathcal{U}$ where $|S| = m$ such that all of $S$ hashes to same slot. Ooops.
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Lesson: For every hash function there is a very bad set. Bad set. Bad.
How many hash functions are there, anyway?

Let $\mathcal{H}$ be the set of all functions from $\mathcal{U} = \{1, \ldots, U\}$ to $\{1, \ldots, m\}$. The number of functions in $\mathcal{H}$ is

(A) $U + m$.
(B) $Um$.
(C) $U^m$.
(D) $m^U$.
(E) $\binom{U+m}{m}$.
(F) The answer is blowing in the wind.
How many bits one need?

Let $\mathcal{H}$ be a set of functions from $\mathcal{U} = \{1, \ldots, U\}$ to $\{1, \ldots, m\}$. Specifying a function in $\mathcal{H}$ requires:

(A) $O(U + m)$ bits.
(B) $O(Um)$ bits.
(C) $O(U^m)$ bits.
(D) $O(m^U)$ bits.
(E) $O(\log |\mathcal{H}|)$ bits.
(F) Many many bits. At least two.
Hash function are often chosen in an ad hoc fashion. Implicit assumption is that input behaves well.

May work well for aircraft control. Susceptible to denial of service attack in routing.
Picking a hash function

1. Hash function are often chosen in an ad hoc fashion. Implicit assumption is that input behaves well.
2. May work well for aircraft control. Susceptible to denial of service attack in routing.

Parameters: $N = |U|$, $m = |T|$, $n = |S|$

1. $\mathcal{H}$ is a family of hash functions: each function $h \in \mathcal{H}$ should be efficient to evaluate (that is, to compute $h(x)$).
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1. \( \mathcal{H} \) is a family of hash functions: each function \( h \in \mathcal{H} \) should be efficient to evaluate (that is, to compute \( h(x) \)).
2. \( h \) is chosen randomly from \( \mathcal{H} \) (typically uniformly at random). Implicitly assumes that \( \mathcal{H} \) allows an efficient sampling.
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1. \( \mathcal{H} \) is a family of hash functions: each function \( h \in \mathcal{H} \) should be efficient to evaluate (that is, to compute \( h(x) \)).
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3. Randomized guarantee: should have the property that for any fixed set \( S \subseteq U \) of size \( m \) the expected number of collisions for a function chosen from \( \mathcal{H} \) should be “small”. Here the expectation is over the randomness in choice of \( h \).
Picking a hash function

**Question:** Why not let \( \mathcal{H} \) be the set of all functions from \( \mathcal{U} \) to \( \{0, 1, \ldots, m - 1\} \)?

Too many functions! A random function has high complexity!

\[
\text{# of functions: } M = m^{|\mathcal{U}|}.
\]

Bits to encode such a function \( \approx \log M = |\mathcal{U}| \log m \).

**Question:** Are there good and compact families \( \mathcal{H} \)?

*Yes... But what it means for \( \mathcal{H} \) to be good and compact.*
Picking a hash function

**Question:** Why not let $\mathcal{H}$ be the set of *all* functions from $\mathcal{U}$ to $\{0, 1, \ldots, m - 1\}$?

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- Yes... But what it means for $\mathcal{H}$ to be good and compact.
Uniform hashing

**Question:** What are good properties of $H$ in distributing data?

1. Consider any element $x \in U$. If $h \in H$ is picked randomly then $x$ should go into a random slot in $T$. In other words $\Pr[h(x) = i] = 1/m$ for every $0 \leq i < m$. (Uniform)

2. Consider any two distinct elements $x, y \in U$. Then if $h \in H$ is picked randomly then the probability of a collision between $x$ and $y$ should be at most $1/m$. In other words $\Pr[h(x) = h(y)] = 1/m$ (cannot be smaller).

3. Second property is stronger than the first and the crucial issue.

**Definition:** A family of hash function $H$ is $(2-)universal$ if for all distinct $x, y \in U$, $\Pr[h(x) = h(y)] = 1/m$ where $m$ is the table size.

Note: The set of all hash functions satisfies stronger properties!
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A family of hash function \( H \) is *(2-)universal* if for all distinct \( x, y \in U \),

\[
\Pr_h[h(x) = h(y)] = \frac{1}{m}
\]

where \( m \) is the table size.

**Note:** The set of all hash functions satisfies stronger properties!
Analyzing Universal Hashing

1. \( T \) is hash table of size \( m \).
2. \( S \subseteq \mathcal{U} \) is a fixed set of size \( \leq m \).
3. \( h \) is chosen randomly from a universal hash family \( \mathcal{H} \).
4. \( x \) is a fixed element of \( \mathcal{U} \).

**Question:** What is the expected time to look up \( x \) in \( T \) using \( h \) assuming chaining used to resolve collisions?
Analyzing Universal Hashing

**Question:** What is the *expected* time to look up \( x \) in \( T \) using \( h \) assuming chaining used to resolve collisions?

1. The time to look up \( x \) is the size of the list at \( T[h(x)] \): same as the number of elements in \( S \) that collide with \( x \) under \( h \).
2. Let \( \ell(x) \) be this number. We want \( E[\ell(x)] \).
3. For \( y \in S \) let \( A_y \) be the event that \( x, y \) collide and \( D_y \) be the corresponding indicator variable.
Number of elements colliding with $x$: $\ell(x) = \sum_{y \in S} D_y$.

$$\Rightarrow E[\ell(x)] = \sum_{y \in S} E[D_y] \quad \text{linearity of expectation}$$

$$= \sum_{y \in S} Pr[h(x) = h(y)]$$

$$= \sum_{y \in S} \frac{1}{m} \quad \text{(since } \mathcal{H} \text{ is a universal hash family)}$$

$$= \frac{|S|}{m}$$

$$= \frac{n}{m}$$

$$\leq 1 \quad \text{(if } |S| \leq m)$$
**Question:** What is the *expected* time to look up $x$ in $T$ using $h$ assuming chaining used to resolve collisions?

**Answer:** $O(n/m)$. 
Analyzing Universal Hashing

**Question:** What is the *expected* time to look up $x$ in $T$ using $h$ assuming chaining used to resolve collisions?

**Answer:** $O(n/m)$.

**Comments:**

1. *$O(1)$* expected time also holds for insertion.
2. Analysis assumes static set $S$ but holds as long as $S$ is a set formed with at most $O(m)$ insertions and deletions.
3. **Worst-case:** look up time can be large! How large? $\Omega(\log n / \log \log n)$
   [Lower bound holds even under stronger assumptions.]
Universal Hash Family

Universal: \( \mathcal{H} \) such that \( \Pr[h(x) = h(y)] = 1/m \).

All functions

\( \mathcal{H} : \) Set of all possible functions \( h : \mathcal{U} \rightarrow \{0, \ldots, m - 1\} \).

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- \( |\mathcal{H}| = m^{|\mathcal{U}|} \)
- representing \( h \) requires \( |\mathcal{U}| \log m \) – Not \( O(1) \)!
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We need \textit{compactly representable} universal family.
Compact Universal Hash Family

Parameters: \( N = |U|, m = |T|, n = |S| \)

1. Choose a prime number \( p > N \).
2. Define function \( h_{a,b}(x) = ((ax + b) \mod p) \mod m \).
Compact Universal Hash Family

Parameters:  \( N = |U|, \ m = |T|, \ n = |S| \)

1. Choose a prime number \( p > N \).
2. Define function \( h_{a,b}(x) = ((ax + b) \mod p) \mod m \).
3. Let \( \mathcal{H} = \{ h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0 \} \ (\mathbb{Z}_p = \{0, 1, \ldots, p - 1\}) \).

Note that \( |\mathcal{H}| = p(p - 1) \).

Theorem \( \mathcal{H} \) is a universal hash family.

Comments:
1. \( h_{a,b} \) can be evaluated in \( O(1) \) time.
2. Easy to store, i.e., just store \( a, b \). Easy to sample.
Compact Universal Hash Family

Parameters: $N = |\mathcal{U}|$, $m = |\mathcal{T}|$, $n = |\mathcal{S}|$

1. Choose a **prime** number $p > N$.
2. Define function $h_{a,b}(x) = ((ax + b) \mod p) \mod m$.
3. Let $\mathcal{H} = \{h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0\}$ ($\mathbb{Z}_p = \{0, 1, \ldots, p - 1\}$). Note that $|\mathcal{H}| = p(p - 1)$. 

Theorem $\mathcal{H}$ is a universal hash family.

Comments:
1. $h_{a,b}$ can be evaluated in $O(1)$ time.
2. Easy to store, i.e., just store $a$, $b$. Easy to sample.
Compact Universal Hash Family

Parameters: \( N = |\mathcal{U}|, \ m = |\mathcal{T}|, \ n = |\mathcal{S}| \)

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\( \mathcal{H} \) is a universal hash family.
Compact Universal Hash Family

Parameters: \( N = |U|, \ m = |T|, \ n = |S| \)

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**Theorem**

\( \mathcal{H} \) is a **universal hash family**.

**Comments:**

1. \( h_{a,b} \) can be evaluated in \( O(1) \) time.
2. Easy to store, i.e., just store \( a, b \). Easy to sample.
Lemma (LemmaUnique)

Let $p$ be a prime number,

$x$: an integer number in $\{1, \ldots, p - 1\}$.

$\implies$ There exists a unique $y$ s.t. $xy = 1 \mod p$.

In other words: For every element there is a unique inverse.

$\implies \mathbb{Z}_p = \{0, 1, \ldots, p - 1\}$ when working modulo $p$ is a field.
Proof of LemmaUnique

Claim

Let $p$ be a prime number. For any $x, y, z \in \{1, \ldots, p - 1\}$ s.t. $y \neq z$, we have that $xy \mod p \neq xz \mod p$.
Proof of LemmaUnique

Claim

Let $p$ be a prime number. For any $x, y, z \in \{1, \ldots, p - 1\}$ s.t. $y \neq z$, we have that $xy \mod p \neq xz \mod p$.

Proof.

Assume for the sake of contradiction $xy \mod p = xz \mod p$. Then

\[
x(y - z) = 0 \mod p
\]

\[
\implies p \text{ divides } x(y - z)
\]

\[
\implies p \text{ divides } y - z
\]

\[
\implies y - z = 0 \implies y = z
\]

And that is a contradiction.
Lemma (LemmaUnique)

Let $p$ be a prime number, $x$: an integer number in $\{1, \ldots, p - 1\}$.

$\implies$ There exists a unique $y$ s.t. $xy = 1 \mod p$.

Proof.

By the above claim if $xy = 1 \mod p$ and $xz = 1 \mod p$ then $y = z$. Hence uniqueness follows.
Proof of LemmaUnique

**Lemma (LemmaUnique)**

Let $p$ be a prime number, $x$: an integer number in $\{1, \ldots, p - 1\}$. 

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**Existence.** For any $x \in \{1, \ldots, p - 1\}$ we have that 

$\{x \times 1 \mod p, x \times 2 \mod p, \ldots, x \times (p - 1) \mod p\} = $
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Existence. For any $x \in \{1, \ldots, p - 1\}$ we have that 
$\{x \ast 1 \mod p, x \ast 2 \mod p, \ldots, x \ast (p - 1) \mod p\} = \{1, 2, \ldots, p - 1\}$. 
$\implies$ There exists a number $y \in \{1, \ldots, p - 1\}$ such that 
$xy = 1 \mod p$.  \hfill \square
Proof of the Theorem: Outline

\[ h_{a,b}(x) = ((ax + b) \mod p) \mod m \].

**Theorem**

\[ \mathcal{H} = \{ h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0 \} \] is universal.

**Proof.**

Fix \( x, y \in \mathcal{U} \). We need to show that 
\[
\Pr_{h_{a,b} \sim \mathcal{H}}[h_{a,b}(x) = h_{a,b}(y)] \leq 1/m.
\] Note that \( |\mathcal{H}| = p(p - 1) \).
Proof of the Theorem: Outline

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**Theorem**

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Note that \( |\mathcal{H}| = p(p-1) \).

1. Let \((a, b)\) (equivalently \(h_{a,b}\)) be *bad* for \(x, y\) if \(h_{a,b}(x) = h_{a,b}(y)\).

2. **Claim:** Number of bad \((a, b)\) is at most \(p(p-1)/m\).

3. Total number of hash functions is \(p(p-1)\) and hence probability of a collision is \(\leq 1/m\).
Intuition for the Claim

\[ g_{a,b}(x) = (ax + b) \mod p, \quad h_{a,b}(x) = (g_{a,b}(x)) \mod m \]

First map \( x \neq y \) to \( r = g_{a,b}(x) \) and \( s = g_{a,b}(y) \).

\[ r \neq s \ (\text{LemmaUnique}) \]
Intuition for the Claim

$$g_{a,b}(x) = (ax + b) \mod p, \quad h_{a,b}(x) = (g_{a,b}(x)) \mod m$$

First map $$x \neq y$$ to $$r = g_{a,b}(x)$$ and $$s = g_{a,b}(y)$$. 
$$r \neq s$$ (LemmaUnique)

As $$(a, b)$$ varies, $$(r, s)$$ takes all possible $$p(p - 1)$$ values. Since $$(a, b)$$ is picked u.a.r., every value of $$(r, s)$$ has equal probability.
Intuition for the Claim

\[ g_{a,b}(x) = (ax + b) \mod p, \quad h_{a,b}(x) = (g_{a,b}(x)) \mod m \]
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Intuition for the Claim

\( g_{a,b}(x) = (ax + b) \pmod{p}, \quad h_{a,b}(x) = (g_{a,b}(x)) \pmod{m} \)

1. First part of mapping maps \((x, y)\) to a random location \((g_{a,b}(x), g_{a,b}(y))\) in the “matrix”.

2. \((g_{a,b}(x), g_{a,b}(y))\) is not on main diagonal.

3. All blue locations are “bad” – map by \( \text{mod } m \) to a location of collision.

4. But... at most \(1/m\) fraction of allowable locations in the matrix are bad.
We need
to show at most $1/m$ fraction of bad $h_{a,b}$

$$h_{a,b}(x) = (((ax + b) \mod p) \mod m)$$

2 lemmas ...

Fix $x \neq y \in \mathbb{Z}_p$, and let $r = (ax + b) \mod p$ and $s = (ay + b) \mod p$.
We need to show at most $\frac{1}{m}$ fraction of bad $h_{a,b}$

$$h_{a,b}(x) = (((ax + b) \mod p) \mod m)$$

2 lemmas ...

Fix $x \neq y \in \mathbb{Z}_p$, and let $r = (ax + b) \mod p$ and $s = (ay + b) \mod p$.

1-to-1 correspondence between $p(p - 1)$ pairs of $(a, b)$ (equivalently $h_{a,b}$) and $p(p - 1)$ pairs of $(r, s)$. 
We need to show at most $\frac{1}{m}$ fraction of bad $h_{a,b}$

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2 lemmas ...

Fix $x \neq y \in \mathbb{Z}_p$, and let $r = (ax + b) \mod p$ and $s = (ay + b) \mod p$.

1. 1-to-1 correspondence between $p(p - 1)$ pairs of $(a, b)$ (equivalently $h_{a,b}$) and $p(p - 1)$ pairs of $(r, s)$.

2. Out of all possible $p(p - 1)$ pairs of $(r, s)$, at most $\frac{p(p - 1)}{m}$ fraction satisfies $r \mod m = s \mod m$. 
Lemma

If \( x \neq y \) then for any \( a, b \in \mathbb{Z}_p \) such that \( a \neq 0 \), we have

\[
ax + b \mod p \neq ay + b \mod p.
\]
Lemma

If $x \neq y$ then for any $a, b \in \mathbb{Z}_p$ such that $a \neq 0$, we have

$$ax + b \mod p \neq ay + b \mod p.$$ 

Proof.

Suppose not

$$ax + b \mod p = ay + b \mod p \Rightarrow a(x - y) \mod p = 0$$
Some Lemmas

Lemma

If \( x \neq y \) then for any \( a, b \in \mathbb{Z}_p \) such that \( a \neq 0 \), we have

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But, \( a \neq 0 \) and \( (x - y) \neq 0 \).
Some Lemmas

Lemma

If $x \neq y$ then for any $a, b \in \mathbb{Z}_p$ such that $a \neq 0$, we have

$$ax + b \ mod \ p \neq ay + b \ mod \ p.$$ 

Proof.

Suppose not

$$ax + b \ mod \ p = ay + b \ mod \ p \Rightarrow a(x - y) \ mod \ p = 0$$

But, $a \neq 0$ and $(x - y) \neq 0$. And $a$ and $(x - y)$ cannot divide $p$ since $p$ is prime and $a < p$ and $(x - y) < p$. Contradiction!
Lemma

If $x \neq y$ then for each $(r, s)$ such that $r \neq s$ and $0 \leq r, s \leq p - 1$ there is exactly one $a, b$ such that $ax + b \mod p = r$ and $ay + b \mod p = s$.

Proof.

Solve the two equations:

$$ax + b = r \mod p \quad \text{and} \quad ay + b = s \mod p$$
Some Lemmas

Lemma

If \( x \neq y \) then for each \((r, s)\) such that \( r \neq s \) and \( 0 \leq r, s \leq p - 1 \) there is exactly one \( a, b \) such that
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Proof.

Solve the two equations:
\[
ax + b = r \mod p \quad \text{and} \quad ay + b = s \mod p.
\]

We get \( a = \frac{r-s}{x-y} \mod p \) and \( b = r - ax \mod p \).

One-to-one correspondence between \((a, b)\) and \((r, s)\).
Understanding the hashing

Once we fix $a$ and $b$, and we are given a value $x$, we compute the hash value of $x$ in two stages:

1. **Compute**: $r \leftarrow (ax + b) \mod p$.
2. **Fold**: $r' \leftarrow r \mod m$

**Collision...**

Given two distinct values $x$ and $y$ they might collide only because of folding.

**Lemma**

$\#$ not equal pairs $(r, s)$ of $\mathbb{Z}_p \times \mathbb{Z}_p$ that are folded to the same number is $p(p - 1)/m$. 
Folding numbers

Lemma

\# pairs \((r, s) \in \mathbb{Z}_p \times \mathbb{Z}_p\) such that \(r \neq s\) and \(r \mod m = s \mod m\) (folded to the same number) is \(p(p - 1)/m\).

Proof.

Consider a pair \((r, s) \in \{0, 1, \ldots, p - 1\}^2\) s.t. \(r \neq s\). Fix \(r\):

1. \(a = r \mod m\).
Folding numbers

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Proof.

Consider a pair \((r, s) \in \{0, 1, \ldots, p - 1\}^2\) s.t. \(r \neq s\). Fix \(r\):

1. \(a = r \mod m\).
2. There are \(\lceil p/m \rceil\) values of \(s\) that fold into \(a\). That is
   \[ r \mod m = s \mod m. \]
3. One of them is when \(r = s\).
4. \(\implies\) \# of colliding pairs
Folding numbers

Lemma

\# pairs \((r, s) \in \mathbb{Z}_p \times \mathbb{Z}_p\) such that \(r \neq s\) and \(r \mod m = s \mod m\) (folded to the same number) is \(p(p-1)/m\).

Proof.

Consider a pair \((r, s) \in \{0, 1, \ldots, p-1\}^2\) s.t. \(r \neq s\). Fix \(r\):

1. \(a = r \mod m\).
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   \[ r \mod m = s \mod m. \]
3. One of them is when \(r = s\).
4. \(\Rightarrow \) \# of colliding pairs \((\lceil p/m \rceil - 1)p \leq (p - 1)p/m\)
Proof of Claim

# of bad pairs is $p(p - 1)/m$

Proof.

Let $a, b \in \mathbb{Z}_p$ such that $a \neq 0$ and $h_{a,b}(x) = h_{a,b}(y)$.

1. Let $r = ax + b \mod p$ and $s = ay + b \mod p$.

2. Collision if and only if $r \mod m = s \mod m$.

3. (Folding error): Number of pairs $(r, s)$ such that $r \neq s$ and $0 \leq r, s \leq p - 1$ and $r \mod m = s \mod m$ is $p(p - 1)/m$.

4. From previous lemma there is one-to-one correspondence between $(a, b)$ and $(r, s)$. Hence total number of bad $(a, b)$ pairs is $p(p - 1)/m$. 

Proof of Claim

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Proof.

Let $a, b \in \mathbb{Z}_p$ such that $a \neq 0$ and $h_{a,b}(x) = h_{a,b}(y)$.

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4. From previous lemma there is one-to-one correspondence between $(a, b)$ and $(r, s)$. Hence total number of bad $(a, b)$ pairs is $p(p - 1)/m$.

Prob of $x$ and $y$ to collide: $\frac{\# \text{ bad } (a, b) \text{ pairs}}{\#(a, b) \text{ pairs}} = \frac{p(p-1)/m}{p(p-1)} = \frac{1}{m}$. 

So far we assumed fixed $S$ of size $\sim m$.

**Question:** What happens as items are inserted and deleted?

1. If $|S|$ grows to more than $cm$ for some constant $c$ then hash table performance clearly degrades.

2. If $|S|$ stays around $\sim m$ but incurs many insertions and deletions then the initial random hash function is no longer random enough!

**Solution:** Rebuild hash table periodically!

1. Choose a new table size based on current number of elements in table.
2. Choose a new random hash function and rehash the elements.
3. Discard old table and hash function.

**Question:** When to rebuild? How expensive?
Rehashing, amortization and...
... making the hash table dynamic

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**Question:** When to rebuild? How expensive?
Rebuilding the hash table

1. Start with table size $m$ where $m$ is some estimate of $|S|$ (can be some large constant).

2. If $|S|$ grows to more than twice current table size, build new hash table (choose a new random hash function) with double the current number of elements. Can also use similar trick if table size falls below quarter the size.

3. If $|S|$ stays roughly the same but more than $c|S|$ operations on table for some chosen constant $c$ (say 10), rebuild.

The amortize cost of rebuilding to previously performed operations. Rebuilding ensures $O(1)$ expected analysis holds even when $S$ changes. Hence $O(1)$ expected look up/insert/delete time dynamic data dictionary data structure!