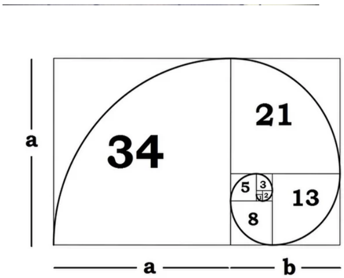
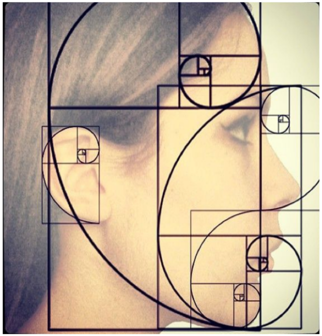
Aside: Golden Ratio

Golden Ratio: A universal law.





Golden ratio $\phi = \lim_{n\to\infty} \frac{a_n+b_n}{a_n} = \frac{1+\sqrt{5}}{2}$

$a_{n+1} = a_n + b_n, \quad b_n = a_{n-1}$

Dynamic Programming I

Lecture 3
Jan 23, 2018

Most slides are courtesy Prof. Chekuri

Recursion

Reduction:

Reduce one problem to another

Recursion

A special case of reduction

1. reduce problem to a *smaller* instance of *itself*
2. self-reduction

1. Problem instance of size n is reduced to one or more instances of size $n - 1$ or less.
2. For termination, problem instances of small size are solved by some other method as **base cases**.

Recursion in Algorithm Design

1. **Tail Recursion**: problem reduced to a *single* recursive call after some work. Easy to convert algorithm into iterative or greedy algorithms.

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   Examples: Merge/Quick Sort, FFT

Recursion in Algorithm Design

1. **Tail Recursion**: problem reduced to a *single* recursive call after some work. Easy to convert algorithm into iterative or greedy algorithms.
2. **Divide and Conquer**: Problem reduced to multiple **independent** sub-problems that are solved separately. Conquer step puts together solution for bigger problem.

   Examples: Merge/Quick Sort, FFT
3. **Dynamic Programming**: problem reduced to multiple (typically) *dependent or overlapping* sub-problems. Use **memoization** to avoid recomputation of common solutions.

Part I

Recursion and Memoization

Recursion, recursion Tree and dependency graph

```
foo(instance  $X$)
    If $X$ is a base case then
        solve it and return solution
    Else
        do some computation
        foo($X_1$)
        do some computation
        foo($X_2$)
        foo($X_3$)
        more computation
        Output solution for $X$
```

Recursion, recursion Tree and dependency graph

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```

Two objects of interest when analyzing **foo**(X)

- *recursion tree* of the recursive implementation
- a *DAG* representing the dependency graph of the distinct subproblems

Fibonacci Numbers

Fibonacci (1200 AD), Pingala (200 BCE).

Numbers defined by recurrence:

$$F(n) = F(n-1) + F(n-2) \text{ and } F(0) = 0, F(1) = 1.$$

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$$F(n) = F(n-1) + F(n-2) \text{ and } F(0) = 0, F(1) = 1.$$

These numbers have many interesting and amazing properties.
A journal *The Fibonacci Quarterly*!

1. $F(n) = (\phi^n - (1-\phi)^n)/\sqrt{5}$ where ϕ is the golden ratio $(1+\sqrt{5})/2 \simeq 1.618$.
2. $\lim_{n\to\infty} F(n+1)/F(n) = \phi$

Recursive Algorithm for Fibonacci Numbers

Question: Given n, compute $F(n)$.

```
Fib($n$):
    if ($n = 0$)
        return 0
    else if ($n = 1$)
        return 1
    else
        return Fib($n - 1$) + Fib($n - 2$)
```

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Running time? Let $T(n)$ be the number of additions in Fib(n).

$$T(n) = T(n-1) + T(n-2) + 1 \text{ and } T(0) = T(1) = 0$$

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Running time? Let $T(n)$ be the number of additions in Fib(n).

$$T(n) = T(n-1) + T(n-2) + 1 \text{ and } T(0) = T(1) = 0$$

Roughly same as $F(n)$

$$T(n) = \Theta(\phi^n)$$

The number of additions is exponential in n.

Recursion tree vs dependency graph

Fib(**5**)

An iterative algorithm for Fibonacci numbers

```
FibIter($n$):
    if ($n = 0$) then
        return 0
    if ($n = 1$) then
        return 1
    $F[0] = 0$
    $F[1] = 1$
```

An iterative algorithm for Fibonacci numbers

```
FibIter($n$):
    if ($n = 0$) then
        return 0
    if ($n = 1$) then
        return 1
    $F[0] = 0$
    $F[1] = 1$
    for $i$ = 2 to $n$ do
        $F[i] = F[i-1] + F[i-2]$
    return $F[n]$
```

Running time: $O(n)$ additions.

What is the difference?

1. Recursive algorithm is recomputing same subproblems many time.
2. Iterative algorithm is computing the value of a subproblem only once by storing them: Memoization.

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Dynamic Programming:

Finding a recursion that can be *effectively/efficiently* memoized.

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.
Every recursive algorithm can be memoized by working with the dependency graph.

Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

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```
Fib($n$):
    if ($n = 0$)
        return $0$
    if ($n = 1$)
        return $1$
    if (Fib($n$) was previously computed)
        return stored value of Fib(n)
    else
        return Fib($n - 1$) + Fib($n - 2$)
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```

How do we keep track of previously computed values?
Two methods: explicitly and implicitly (via data structure)

Automatic explicit memoization

Initialize table/array M of size n such that $M[i] = -1$ for $i = 0, \ldots, n$.

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```
Fib($n$):
    if ($n = 0$)
        return 0
    if ($n = 1$)
        return 1
    if ($M[n] \neq -1$) (* $M[n]$ has stored value of Fib($n$) *)
        return $M[n]$
    $M[n] \leftarrow$ Fib($n-1$) + Fib($n-2$)
    return $M[n]$
```

To allocate memory need to know upfront the number of subproblems for a given input size n

Automatic implicit memoization

Initialize a (dynamic) dictionary data structure D to empty

```
Fib($n$):
    if ($n = 0$)
        return 0
    if ($n = 1$)
        return 1
    if ($n$ is already in $D$)
        return value stored with $n$ in $D$
    $val \leftarrow$ Fib($n-1$) + Fib($n-2$)
    Store $(n, val)$ in $D$
    return $val$
```

Explicit vs Implicit Memoization

1. Explicit memoization or iterative algorithm preferred if one can analyze problem ahead of time. Allows for efficient memory allocation and access.

Explicit vs Implicit Memoization

1. Explicit memoization or iterative algorithm preferred if one can analyze problem ahead of time. Allows for efficient memory allocation and access.
2. Implicit and automatic memoization used when problem structure or algorithm is either not well understood or in fact unknown to the underlying system.
    1. Need to pay overhead of data-structure.
    2. Functional languages such as LISP automatically do memoization, usually via hashing based dictionaries.

Back to Fibonacci Numbers

Saving space. Do we need an array of n numbers?

Back to Fibonacci Numbers

Saving space. Do we need an array of n numbers? Not really.

```
FibIter($n$):
    if ($n = 0$) then
        return 0
    if ($n = 1$) then
        return 1
    $prev2 = 0$
    $prev1 = 1$
    for $i$ = 2 to $n$ do
        $temp = prev1 + prev2$
        $prev2 = prev1$
        $prev1 = temp$

    return $prev1$
```

What is Dynamic Programming?

Every recursion can be memoized. Automatic memoization does not help us understand whether the resulting algorithm is efficient or not.

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- Given a recursive algorithm, how do we analyze complexity when it is memoized?

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- How do we recognize whether a problem admits a dynamic programming based efficient algorithm?

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A recursion that when memoized leads to an *efficient* algorithm.

Key Questions:

- Given a recursive algorithm, how do we analyze complexity when it is memoized?
- How do we recognize whether a problem admits a dynamic programming based efficient algorithm?
- How do we further optimize time and space of a dynamic programming based algorithm?

Part II

Edit Distance

Edit Distance

Definition

Edit distance between two words X and Y is the number of letter insertions, letter deletions and letter substitutions required to obtain Y from X.

Example

The edit distance between FOOD and MONEY is at most **4**:

$$\mathbf{\underline{F}OOD} \rightarrow \mathbf{MO\underline{O}D} \rightarrow \mathbf{MON\underline{O}D} \rightarrow \mathbf{MONE\underline{D}} \rightarrow \mathbf{MONEY}$$

Edit Distance: Alternate View

Alignment

Place words one on top of the other, with gaps in the first word indicating insertions, and gaps in the second word indicating deletions.

$$\begin{array}{ccccc} \mathbf{F} & \mathbf{O} & \mathbf{O} & & \mathbf{D} \\ \mathbf{M} & \mathbf{O} & \mathbf{N} & \mathbf{E} & \mathbf{Y} \end{array}$$

Edit Distance: Alternate View

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Formally, an alignment is a sequence M of pairs (i,j) such that each index appears exactly once, and there is no "crossing": if $(i,j),\ldots,(i',j')$ then $i < i'$ and $j < j'$. One of i or j could be empty, in which case no comparision. In the above example, this is $M = \{(1,1),(2,2),(3,3),(\ ,4),(4,5)\}$.

Edit Distance: Alternate View

Alignment

Place words one on top of the other, with gaps in the first word indicating insertions, and gaps in the second word indicating deletions.

F	O	O		D
M	O	N	E	Y

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Cost of an alignment: the number of mismatched columns.

Edit Distance Problem

Problem

Given two words, find the edit distance between them, i.e., an alignment of smallest cost.

Edit Distance

Basic observation

Let $X = \alpha x$ and $Y = \beta y$
α, β: strings. x and y single characters.

Possible alignments between X and Y

α	x
β	y

or

α	x
βy	

or

αx	
β	y

Edit Distance

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Prefixes must have optimal alignment!

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Possible alignments between X and Y

α	x
β	y

or

α	x
βy	

or

αx	
β	y

Observation

Prefixes must have optimal alignment!

$$EDIST(X, Y) = \min \begin{cases} EDIST(\alpha, \beta) + [x \neq y] \\ 1 + EDIST(\alpha, Y) \\ 1 + EDIST(X, \beta) \end{cases}$$

Recursive Algorithm

Assume X is stored in array $A[1..m]$ and Y is stored in $B[1..n]$

```
EDIST(A[1..m], B[1..n])
    If ($m = 0$) return $n$
    If ($n = 0$) return $m$
```

Recursive Algorithm

Assume X is stored in array $A[1..m]$ and Y is stored in $B[1..n]$

```
$EDIST(A[1..m], B[1..n])$
    If ($m = 0$) return $n$
    If ($n = 0$) return $m$
    $m_1 = 1 + EDIST(A[1..(m-1)], B[1..n])$
    $m_2 = 1 + EDIST(A[1..m], B[1..(n-1)]))$
    If ($A[m] = B[n]$) then
        $m_3 = EDIST(A[1..(m-1)], B[1..(n-1)])$
    Else
        $m_3 = 1 + EDIST(A[1..(m-1)], B[1..(n-1)])$
    return $\min(m_1, m_2, m_3)$
```

Example

DEED and DREAD

Subproblems and Recurrence

Each subproblem corresponds to a prefix of X and a prefix of Y

Optimal Costs

Let $\mathrm{Opt}(i, j)$ be optimal cost of aligning $x_1 \cdots x_i$ and $y_1 \cdots y_j$. Then

$$\mathrm{Opt}(i, j) = \min \begin{cases} [x_i \neq y_j] + \mathrm{Opt}(i-1, j-1), \\ 1 + \mathrm{Opt}(i-1, j), \\ 1 + \mathrm{Opt}(i, j-1) \end{cases}$$

Base Cases: $\mathrm{Opt}(i, 0) = i$ and $\mathrm{Opt}(0, j) = j$

Memoizing the Recursive Algorithm

```
$\textit{int}$   $M[0..m][0..n]$
Initialize all entries of $M[i][j]$ to $\infty$
return $EDIST(A[1..m], B[1..n])$
```

Memoizing the Recursive Algorithm

```
$int$   $M[0..m][0..n]$
Initialize all entries of $M[i][j]$ to $\infty$
return $EDIST(A[1..m], B[1..n])$
```

```
$EDIST(A[1..m], B[1..n])$
    If $(M[i][j] < \infty)$ return $M[i][j]$    (* return stored value *)
    If $(m = 0)$
        $M[i][j] = n$
    ElseIf $(n = 0)$
        $M[i][j] = m$
```

Memoizing the Recursive Algorithm

```
$int \quad M[0..m][0..n]$
Initialize all entries of $M[i][j]$ to $\infty$
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```
$EDIST(A[1..m], B[1..n])$
    If $(M[i][j] < \infty)$ return $M[i][j]$   (* return stored value *)
    If $(m = 0)$
        $M[i][j] = n$
    ElseIf $(n = 0)$
        $M[i][j] = m$
    Else
        $m_1 = 1 + EDIST(A[1..(m-1)], B[1..n])$
        $m_2 = 1 + EDIST(A[1..m], B[1..(n-1)]))$
        If $(A[m] = B[n])$ $m_3 = EDIST(A[1..(m-1)], B[1..(n-1)])$
        Else $m_3 = 1 + EDIST(A[1..(m-1)], B[1..(n-1)])$
        $M[i][j] = \min(m_1, m_2, m_3)$
    return $M[i][j]$
```

Matrix and DAG of Computation

Matrix M:

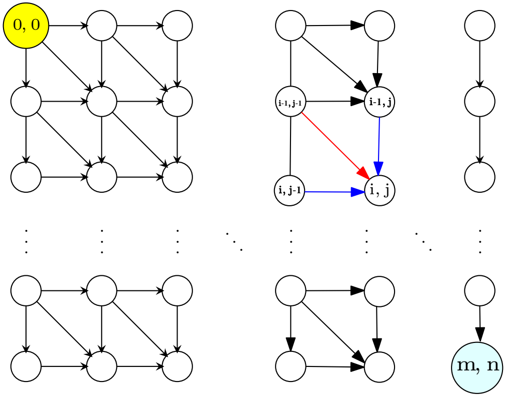

Figure: Dependency of matrix entries in the recursive algorithm of previous slide

Removing Recursion to obtain Iterative Algorithm

```
$EDIST(A[1..m], B[1..n])$
    $int \quad M[0..m][0..n]$
    for $i = 1$ to $m$ do $M[i, 0] = i$
    for $j = 1$ to $n$ do $M[0, j] = j$

    for $i = 1$ to $m$ do
        for $j = 1$ to $n$ do
            $M[i][j] = \min \begin{cases} [x_i = y_j] + M[i-1][j-1], \\ 1 + M[i-1][j], \\ 1 + M[i][j-1] \end{cases}$
```

Removing Recursion to obtain Iterative Algorithm

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$EDIST(A[1..m], B[1..n])$
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```

Analysis

Removing Recursion to obtain Iterative Algorithm

```
$EDIST(A[1..m], B[1..n])$
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```

Analysis

1. Running time is $O(mn)$.
2. Space used is $O(mn)$.

Matrix and DAG of Computation (revisited)

Matrix M:

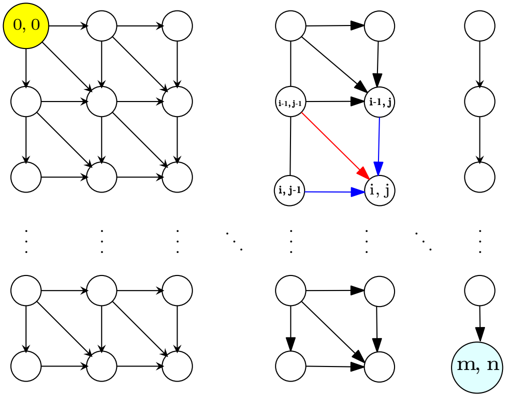

Figure: Iterative algorithm in previous slide computes values in row order.

Finding an Optimum Solution

The DP algorithm finds the minimum edit distance in $O(nm)$ space and time.

Question: Can we find a specific alignment which achieves the minimum?

Finding an Optimum Solution

The DP algorithm finds the minimum edit distance in $O(nm)$ space and time.

Question: Can we find a specific alignment which achieves the minimum?

Exercise: Show that one can find an optimum solution after computing the optimum value. Key idea is to store back pointers when computing $Opt(i, j)$ to know how we calculated it. See notes for more details.

Dynamic Programming Template

1. Come up with a recursive algorithm to solve problem
2. Understand the structure/number of the subproblems generated by recursion
3. Memoize the recursion $\rightarrow$ DP
    - set up compact notation for subproblems
    - set up a data structure for storing subproblems
4. Iterative algorithm
    - Understand dependency graph on subproblems
    - Pick an evaluation order (any topological sort of the dependency dag)
5. Analyze time and space
6. Optimize

Part III

Knapsack

Knapsack Problem

Input Given a Knapsack of capacity W lbs. and n objects with ith object having weight w_i and value v_i; assume W, w_i, v_i are all positive integers

Goal Fill the Knapsack without exceeding weight limit while maximizing value.

Knapsack Problem

Input	Given a Knapsack of capacity W lbs. and n objects with ith object having weight w_i and value v_i; assume W, w_i, v_i are all positive integers
Goal	Fill the Knapsack without exceeding weight limit while maximizing value.

Basic problem that arises in many applications as a sub-problem.

Knapsack Example

Example

Item	I_1	I_2	I_3	I_4	I_5
Value	1	6	18	22	28
Weight	1	2	5	6	7

If $W = \mathbf{11}$, the best is $\{I_3, I_4\}$ giving value 40.

Knapsack Example

Example

Item	I_1	I_2	I_3	I_4	I_5
Value	1	6	18	22	28
Weight	1	2	5	6	7

If $W = 11$, the best is $\{I_3, I_4\}$ giving value 40.

Special Case

When $v_i = w_i$, the Knapsack problem is called the Subset Sum Problem.

Knapsack

For the following instance of Knapsack:

Item	I_1	I_2	I_3	I_4	I_5
Value	1	6	16	22	28
Weight	1	2	5	6	7

and weight limit $W = 15$. The best solution has value:

- **(A)** 22
- **(B)** 28
- **(C)** 38
- **(D)** 50
- **(E)** 56

Greedy Approach

1. Pick objects with greatest value
    1. Let $W = 2$, $w_1 = w_2 = 1$, $w_3 = 2$, $v_1 = v_2 = 2$ and $v_3 = 3$;

Greedy Approach

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    1. Let $W = 2$, $w_1 = 1$, $w_2 = 2$, $v_1 = 1$ and $v_2 = 3$;greedy strategy will pick $\{1\}$, but the optimal is $\{2\}$
3. Pick objects with largest v_i/w_i ratio
    1. Let $W = 4$, $w_1 = w_2 = 2$, $w_3 = 3$, $v_1 = v_2 = 3$ and $v_3 = 5$;

Greedy Approach

1. Pick objects with greatest value
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   1. Let $W = 4$, $w_1 = w_2 = 2$, $w_3 = 3$, $v_1 = v_2 = 3$ and $v_3 = 5$;greedy strategy will pick $\{3\}$, but the optimal is $\{1, 2\}$
   2. **Aside:** Can show that a slight modification always gives half the optimum profit: pick the better of the output of this algorithm and the largest value item. Also, the algorithms gives better approximations when all item weights are small when compared to W.

Towards a Recursive Algorithms

First guess: $\text{Opt}(i)$ is the optimum solution value for items $1, \ldots, i$.

Observation

Consider an optimal solution $\mathcal{O}$ for $1, \ldots, i$

Case item $i \notin \mathcal{O}$ *$\mathcal{O}$ is an optimal solution to items 1 to $i - 1$*

Case item $i \in \mathcal{O}$ *Then $\mathcal{O} - \{i\}$ is an optimum solution for items 1 to $i - 1$ in knapsack of capacity $W - w_i$.*

Towards a Recursive Algorithms

First guess: $\text{Opt}(i)$ is the optimum solution value for items $1, \ldots, i$.

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$\mathrm{Opt}(i, w)$: optimum profit for items 1 to i in knapsack of size w
Goal: compute $\mathrm{Opt}(n, W)$

Dynamic Programming Solution

Definition

Let $\mathrm{Opt}(i, w)$ be the optimal way of picking items from 1 to i, with total weight not exceeding w.

$$\mathrm{Opt}(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ \mathrm{Opt}(i-1, w) & \text{if } w_i > w \\ \max \begin{cases} \mathrm{Opt}(i-1, w) \\ \mathrm{Opt}(i-1, w - w_i) + v_i \end{cases} & \text{otherwise} \end{cases}$$

Number of subproblem generated by $Opt(n, W)$ is $O(nW)$.

An Iterative Algorithm

```
for $w = 0$ to $W$ do
    $M[0, w] = 0$
for $i = 1$ to $n$ do
    for $w = 1$ to $W$ do
        if ($w_i > w$) then
            $M[i, w] = M[i-1, w]$
        else
            $M[i, w] = \max(M[i-1, w], M[i-1, w-w_i] + v_i)$
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Running Time

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Running Time

1. Time taken is $O(nW)$
2. Input has size $O(n + \log W + \sum_{i=1}^{n}(\log v_i + \log w_i))$; so running time not polynomial but "pseudo-polynomial"!

Introducing a Variable

For the Knapsack problem obtaining a recursive algorithm required introducing a new variable, namely the size of the knapsack.

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How do we figure out when this is possible?

Heuristic answer that works for many problems: Try divide and conquer or obvious recursion: if problem is **not** decomposable then introduce the "information" required to decompose as new variable(s). Will see several examples to make this idea concrete.

Knapsack Algorithm and Polynomial time

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2. Running time of dynamic programming algorithm: $O(nW)$.
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4. Example: $W = 2^n$ and $w_i, v_i \in [1..2^n]$. Input size is $O(n^2)$, running time is $O(n2^n)$ arithmetic/comparisons.
5. Algorithm is called a **pseudo-polynomial** time algorithm because running time is polynomial if *numbers* in input are of size polynomial in the **combinatorial size** of problem.
6. Knapsack is **NP-Hard** if numbers are not polynomial in n.