Aside: Golden Ratio

Golden Ratio: A universal law.

Golden ratio \( \phi = \lim_{n \to \infty} \frac{a_n + b_n}{a_n} = \frac{1 + \sqrt{5}}{2} \)

\( a_{n+1} = a_n + b_n, \quad b_n = a_{n-1} \)
Dynamic Programming I

Lecture 3
Jan 23, 2018

Most slides are courtesy Prof. Chekuri
Recursion

Reduction:
Reduce one problem to another

Recursion
A special case of reduction
1. reduce problem to a *smaller* instance of *itself*
2. self-reduction

1. Problem instance of size \( n \) is reduced to one or more instances of size \( n - 1 \) or less.
2. For termination, problem instances of small size are solved by some other method as *base cases*.
Tail Recursion: problem reduced to a single recursive call after some work. Easy to convert algorithm into iterative or greedy algorithms.
Recursion in Algorithm Design

1. **Tail Recursion**: problem reduced to a single recursive call after some work. Easy to convert algorithm into iterative or greedy algorithms.

2. **Divide and Conquer**: Problem reduced to multiple independent sub-problems that are solved separately. Conquer step puts together solution for bigger problem.

Examples: Merge/Quick Sort, FFT
Recursion in Algorithm Design

1. **Tail Recursion**: problem reduced to a *single* recursive call after some work. Easy to convert algorithm into iterative or greedy algorithms.

2. **Divide and Conquer**: Problem reduced to multiple independent sub-problems that are solved separately. Conquer step puts together solution for bigger problem.
   
   Examples: Merge/Quick Sort, FFT

3. **Dynamic Programming**: problem reduced to multiple (typically) dependent or overlapping sub-problems. Use **memoization** to avoid recomputation of common solutions.
Part I

Recursion and Memoization
Recursion, recursion Tree and dependency graph

```plaintext
foo(instance X)
    If X is a base case then
        solve it and return solution
    Else
        do some computation
        foo(X_1)
        do some computation
        foo(X_2)
        foo(X_3)
        more computation
    Output solution for X
```
Recursion, recursion Tree and dependency graph

foo(instance X)
If X is a base case then
    solve it and return solution
Else
    do some computation
    foo(X_1)
    do some computation
    foo(X_2)
    foo(X_3)
    more computation
Output solution for X

Two objects of interest when analyzing foo(X)
- recursion tree of the recursive implementation
- a DAG representing the dependency graph of the distinct subproblems
Fibonacci (1200 AD), Pingala (200 BCE). Numbers defined by recurrence:

\[ F(n) = F(n - 1) + F(n - 2) \text{ and } F(0) = 0, F(1) = 1. \]
Fibonacci Numbers

Fibonacci (1200 AD), Pingala (200 BCE).
Numbers defined by recurrence:

\[ F(n) = F(n - 1) + F(n - 2) \text{ and } F(0) = 0, F(1) = 1. \]

These numbers have many interesting and amazing properties.
A journal *The Fibonacci Quarterly!*

1. \[ F(n) = \left( \phi^n - (1 - \phi)^n \right) / \sqrt{5} \text{ where } \phi \text{ is the golden ratio } (1 + \sqrt{5})/2 \approx 1.618. \]
2. \[ \lim_{n \to \infty} F(n + 1)/F(n) = \phi \]
Question: Given $n$, compute $F(n)$.

\[
\text{Fib}(n):
\]
\[
\begin{align*}
\text{if } (n = 0) & \text{ return } 0 \\
\text{else if } (n = 1) & \text{ return } 1 \\
\text{else} & \\
\text{return } \text{Fib}(n - 1) + \text{Fib}(n - 2)
\end{align*}
\]
Recursive Algorithm for Fibonacci Numbers

Question: Given $n$, compute $F(n)$.

```python
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        return Fib(n − 1) + Fib(n − 2)
```

Running time? Let $T(n)$ be the number of additions in Fib(n).

$$T(n) = T(n − 1) + T(n − 2) + 1 \text{ and } T(0) = T(1) = 0$$
Recursive Algorithm for Fibonacci Numbers

Question: Given \( n \), compute \( F(n) \).

\[
\text{Fib}(n) :
\begin{align*}
\text{if } (n &= 0) \\
& \quad \text{return } 0 \\
\text{else if } (n &= 1) \\
& \quad \text{return } 1 \\
\text{else} \\
& \quad \text{return } \text{Fib}(n - 1) + \text{Fib}(n - 2)
\end{align*}
\]

Running time? Let \( T(n) \) be the number of additions in Fib(n).

\[
T(n) = T(n - 1) + T(n - 2) + 1 \text{ and } T(0) = T(1) = 0
\]

Roughly same as \( F(n) \)

\[
T(n) = \Theta(\phi^n)
\]

The number of additions is exponential in \( n \).
Recursion tree vs dependency graph

\textbf{Fib}(5)
An iterative algorithm for Fibonacci numbers

\textbf{FibIter}(n):
\begin{align*}
& \text{if } (n = 0) \text{ then return 0} \\
& \text{if } (n = 1) \text{ then return 1} \\
& F[0] = 0 \\
& F[1] = 1
\end{align*}
An iterative algorithm for Fibonacci numbers

\textbf{FibIter}(n):
\begin{align*}
\text{if } (n = 0) \text{ then} & \quad \text{return } 0 \\
\text{if } (n = 1) \text{ then} & \quad \text{return } 1 \\
F[0] & = 0 \\
F[1] & = 1 \\
\text{for } i = 2 \text{ to } n \text{ do} & \quad F[i] = F[i - 1] + F[i - 2] \\
\text{return } F[n]
\end{align*}

Running time: $O(n)$ additions.
What is the difference?

1. Recursive algorithm is recomputing same subproblems many times.

2. Iterative algorithm is computing the value of a subproblem only once by storing them: Memoization.
What is the difference?

1. Recursive algorithm is recomputing same subproblems many times.
2. Iterative algorithm is computing the value of a subproblem only once by storing them: Memoization.

Dynamic Programming:

Finding a recursion that can be effectively/efficiently memoized.

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.
Every recursive algorithm can be memoized by working with the dependency graph.
Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

\[
\text{Fib}(n) = \begin{cases} 
0 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
\text{Fib}(n-1) + \text{Fib}(n-2) & \text{if } \text{Fib}(n) \text{ was previously computed} \\
\text{stored value of Fib}(n) & \text{otherwise} 
\end{cases}
\]

How do we keep track of previously computed values?

Two methods: explicitly and implicitly (via data structure).
Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

\[
\text{Fib}(n) :
\begin{align*}
\text{if } (n = 0) & \quad \text{return } 0 \\
\text{if } (n = 1) & \quad \text{return } 1 \\
\text{if } (\text{Fib}(n) \text{ was previously computed}) & \quad \text{return stored value of Fib}(n) \\
\text{else} & \quad \text{return } \text{Fib}(n - 1) + \text{Fib}(n - 2)
\end{align*}
\]
Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

\[
\text{Fib}(n) : \\
\quad \text{if } (n = 0) \\
\quad \quad \text{return } 0 \\
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\quad \quad \text{return } 1 \\
\quad \text{if } (\text{Fib}(n) \text{ was previously computed}) \\
\quad \quad \text{return stored value of Fib}(n) \\
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How do we keep track of previously computed values?
Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

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&\text{if } (\text{Fib}(n) \text{ was previously computed}) \\
&\quad \text{return stored value of Fib}(n) \\
&\text{else} \\
&\quad \text{return Fib}(n - 1) + \text{Fib}(n - 2)
\end{align*}
\]

How do we keep track of previously computed values?
Two methods: explicitly and implicitly (via data structure)
Automatic explicit memoization

Initialize table/array $M$ of size $n$ such that $M[i] = -1$ for $i = 0, \ldots, n$. 

\[
\text{Fib}(n) :=
\begin{align*}
\text{if } (n = 0) & \quad \text{return } 0 \\
\text{if } (n = 1) & \quad \text{return } 1 \\
\text{if } (M[n] \neq -1) \quad \text{(* } M[n] \text{ has stored value of } \text{Fib}(n) \text{ *)} & \quad \text{return } M[n] \\
M[n] & \leftarrow \text{Fib}(n-1) + \text{Fib}(n-2) \\
\text{return } M[n]
\end{align*}
\]
Automatic explicit memoization

Initialize table/array $M$ of size $n$ such that $M[i] = -1$ for $i = 0, \ldots, n$.

```
Fib(n):
    if (n == 0)
        return 0
    if (n == 1)
        return 1
    if ($M[n] \neq -1$) (* $M[n]$ has stored value of Fib(n) *)
        return $M[n]$
    $M[n] \leftarrow$ Fib($n - 1$) + Fib($n - 2$)
    return $M[n]$
```

To allocate memory need to know upfront the number of subproblems for a given input size $n$
Automatic implicit memoization

Initialize a (dynamic) dictionary data structure \( D \) to empty

\[
\text{Fib}(n) : \\
\text{if } (n = 0) \quad \text{return } 0 \\
\text{if } (n = 1) \quad \text{return } 1 \\
\text{if } (n \text{ is already in } D) \quad \text{return value stored with } n \text{ in } D \\
val \leftarrow \text{Fib}(n - 1) + \text{Fib}(n - 2) \\
\text{Store } (n, val) \text{ in } D \\
\text{return } val
\]
Explicit vs Implicit Memoization

Explicit memoization or iterative algorithm preferred if one can analyze problem ahead of time. Allows for efficient memory allocation and access.
Explicit vs Implicit Memoization

1. Explicit memoization or iterative algorithm preferred if one can analyze problem ahead of time. Allows for efficient memory allocation and access.

2. Implicit and automatic memoization used when problem structure or algorithm is either not well understood or in fact unknown to the underlying system.
   - Need to pay overhead of data-structure.
   - Functional languages such as LISP automatically do memoization, usually via hashing based dictionaries.
Saving space. Do we need an array of $n$ numbers?

```python
FibIter(n):
    if (n == 0):
        return 0
    if (n == 1):
        return 1
    prev2 = 0
    prev1 = 1
    for i = 2 to n:
        temp = prev1 + prev2
        prev2 = prev1
        prev1 = temp
    return prev1
```
Back to Fibonacci Numbers

Saving space. Do we need an array of \( n \) numbers? Not really.

\[
\text{FibIter}\left( n \right):
\begin{align*}
&\quad \text{if } (n = 0) \text{ then} \\
&\quad \quad \text{return } 0 \\
&\quad \text{if } (n = 1) \text{ then} \\
&\quad \quad \text{return } 1 \\
&\quad \text{prev}_2 = 0 \\
&\quad \text{prev}_1 = 1 \\
&\quad \text{for } i = 2 \text{ to } n \text{ do} \\
&\quad \quad \text{temp} = \text{prev}_1 + \text{prev}_2 \\
&\quad \quad \text{prev}_2 = \text{prev}_1 \\
&\quad \quad \text{prev}_1 = \text{temp} \\
&\quad \text{return } \text{prev}_1
\end{align*}
\]
What is Dynamic Programming?

Every recursion can be memoized. Automatic memoization does not help us understand whether the resulting algorithm is efficient or not.
What is Dynamic Programming?

Every recursion can be memoized. Automatic memoization does not help us understand whether the resulting algorithm is efficient or not.

**Dynamic Programming:**
A recursion that when memoized leads to an *efficient* algorithm.
What is Dynamic Programming?

Every recursion can be memoized. Automatic memoization does not help us understand whether the resulting algorithm is efficient or not.

**Dynamic Programming:** A recursion that when memoized leads to an *efficient* algorithm.

**Key Questions:**
- Given a recursive algorithm, how do we analyze complexity when it is memoized?
What is Dynamic Programming?

Every recursion can be memoized. Automatic memoization does not help us understand whether the resulting algorithm is efficient or not.

**Dynamic Programming:**
A recursion that when memoized leads to an *efficient* algorithm.

Key Questions:
- Given a recursive algorithm, how do we analyze complexity when it is memoized?
- How do we recognize whether a problem admits a dynamic programming based efficient algorithm?
What is Dynamic Programming?

Every recursion can be memoized. Automatic memoization does not help us understand whether the resulting algorithm is efficient or not.

Dynamic Programming:
A recursion that when memoized leads to an efficient algorithm.

Key Questions:
- Given a recursive algorithm, how do we analyze complexity when it is memoized?
- How do we recognize whether a problem admits a dynamic programming based efficient algorithm?
- How do we further optimize time and space of a dynamic programming based algorithm?
**Definition**

Edit distance between two words $X$ and $Y$ is the number of letter insertions, letter deletions and letter substitutions required to obtain $Y$ from $X$.

**Example**

The edit distance between FOOD and MONEY is at most 4:

$$
\text{FOOD} \rightarrow \underline{\text{MOOD}} \rightarrow \text{MONOD} \rightarrow \underline{\text{MONED}} \rightarrow \text{MONEY}
$$
Alignment

Place words one on top of the other, with gaps in the first word indicating insertions, and gaps in the second word indicating deletions.

FOOD
MONEY
Alignment

Place words one on top of the other, with gaps in the first word indicating insertions, and gaps in the second word indicating deletions.

\[
\begin{align*}
\text{FOOD} \\
\text{MONEY}
\end{align*}
\]

Formally, an alignment $M$ is a sequence of pairs $(i, j)$ such that each index appears exactly once, and there is no “crossing”: if $(i, j), \ldots, (i', j')$ then $i < i'$ and $j < j'$. One of $i$ or $j$ could be empty, in which case no comparison. In the above example, this is $M = \{(1, 1), (2, 2), (3, 3), (4, 4), (4, 5)\}$.
Alignment

Place words one on top of the other, with gaps in the first word indicating insertions, and gaps in the second word indicating deletions.

FOOOOD
MONEY

Formally, an alignment is a sequence $M$ of pairs $(i, j)$ such that each index appears exactly once, and there is no “crossing”: if $(i, j), \ldots, (i', j')$ then $i < i'$ and $j < j'$. One of $i$ or $j$ could be empty, in which case no comparison. In the above example, this is $M = \{(1, 1), (2, 2), (3, 3), (\ , 4), (4, 5)\}$.

Cost of an alignment: the number of mismatched columns.
Edit Distance Problem

Problem
Given two words, find the edit distance between them, i.e., an alignment of smallest cost.
Edit Distance

Basic observation

Let $X = \alpha x$ and $Y = \beta y$

$\alpha, \beta$: strings. $x$ and $y$ single characters.

Possible alignments between $X$ and $Y$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$x$</th>
<th>or</th>
<th>$\alpha$</th>
<th>$x$</th>
<th>or</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$y$</td>
<td></td>
<td>$\beta$</td>
<td>$y$</td>
<td></td>
</tr>
</tbody>
</table>

Observation Prefixes must have optimal alignment!

$$\text{EDIST}(X, Y) = \min \begin{cases} 
\text{EDIST}(\alpha, \beta) + [x \neq y] + 1 + \text{EDIST}(\alpha, Y) \\
1 + \text{EDIST}(X, \beta) 
\end{cases}$$
Let $X = \alpha x$ and $Y = \beta y$

$\alpha, \beta$: strings. $x$ and $y$ single characters.

Possible alignments between $X$ and $Y$

| $\alpha$ | $x$ | or | $\alpha$ | $x$ | or | $\alpha x$ | |
| --- | --- | or | --- | --- | or | --- | --- |
| $\beta$ | $y$ | | $\beta y$ | | | $\beta$ | $y$ |

**Observation**

*Prefixes must have optimal alignment!*
Let $X = \alpha x$ and $Y = \beta y$

$\alpha, \beta$: strings. $x$ and $y$ single characters.

Possible alignments between $X$ and $Y$

| $\alpha$ | $x$ | or | $\alpha$ | $x$ | or | $\alpha x$ |   |
|----------|-----|    | $\beta$ | $y$  |    | $\beta$ | $y$ |

Observation

*Prefixes must have optimal alignment!*

$$EDIST(X, Y) = \min \begin{cases} EDIST(\alpha, \beta) + [x \neq y] \\ 1 + EDIST(\alpha, Y) \\ 1 + EDIST(X, \beta) \end{cases}$$
Recursive Algorithm

Assume $X$ is stored in array $A[1..m]$ and $Y$ is stored in $B[1..n]$

$$EDIST(A[1..m], B[1..n])$$

If ($m = 0$) return $n$
If ($n = 0$) return $m$
Recursive Algorithm

Assume \( X \) is stored in array \( A[1..m] \) and \( Y \) is stored in \( B[1..n] \)

\[
EDIST(A[1..m], B[1..n])
\]

If \( m = 0 \) return \( n \)
If \( n = 0 \) return \( m \)

\[
m_1 = 1 + EDIST(A[1..(m - 1)], B[1..n])
\]

\[
m_2 = 1 + EDIST(A[1..m], B[1..(n - 1)])
\]

If \( A[m] = B[n] \) then
\[
m_3 = EDIST(A[1..(m - 1)], B[1..(n - 1)])
\]
Else
\[
m_3 = 1 + EDIST(A[1..(m - 1)], B[1..(n - 1)])
\]

return \( \min(m_1, m_2, m_3) \)
Example

DEED and DREAD
Each subproblem corresponds to a prefix of $X$ and a prefix of $Y$

**Optimal Costs**

Let $\text{Opt}(i, j)$ be optimal cost of aligning $x_1 \cdots x_i$ and $y_1 \cdots y_j$.

Then

$$\text{Opt}(i, j) = \min \left\{ \begin{array}{ll}
[x_i \neq y_j] + \text{Opt}(i - 1, j - 1), \\
1 + \text{Opt}(i - 1, j), \\
1 + \text{Opt}(i, j - 1)
\end{array} \right. $$

Base Cases: $\text{Opt}(i, 0) = i$ and $\text{Opt}(0, j) = j$
Memoizing the Recursive Algorithm

\[
\text{int } M[0..m][0..n] \\
\text{Initialize all entries of } M[i][j] \text{ to } \infty \\
\text{return } \text{EDIST}(A[1..m], B[1..n])
\]
Memoizing the Recursive Algorithm

```plaintext
int M[0..m][0..n]
```

Initialize all entries of $M[i][j]$ to $\infty$

```plaintext
return EDIST(A[1..m], B[1..n])
```

**EDIST(A[1..m], B[1..n])**

- If $(M[i][j] < \infty)$ return $M[i][j]$  (* return stored value * )
- If ($m = 0$)
  ```plaintext
  M[i][j] = n
  ```
- ElseIf ($n = 0$)
  ```plaintext
  M[i][j] = m
  ```
Memoizing the Recursive Algorithm

\[
\text{int } M[0..m][0..n] \\
\text{Initialize all entries of } M[i][j] \text{ to } \infty \\
\text{return } \text{EDIST}(A[1..m], B[1..n])
\]

\[
\text{EDIST}(A[1..m], B[1..n]) \\
\text{If } (M[i][j] < \infty) \text{ return } M[i][j] \quad (* \text{ return stored value } *) \\
\text{If } (m = 0) \\
\quad M[i][j] = n \\
\text{ElseIf } (n = 0) \\
\quad M[i][j] = m \\
\text{Else} \\
\quad m_1 = 1 + \text{EDIST}(A[1..(m - 1)], B[1..n]) \\
\quad m_2 = 1 + \text{EDIST}(A[1..m], B[1..(n - 1)]) \\
\text{If } (A[m] = B[n]) \quad m_3 = \text{EDIST}(A[1..(m - 1)], B[1..(n - 1)]) \\
\text{Else } m_3 = 1 + \text{EDIST}(A[1..(m - 1)], B[1..(n - 1)]) \\
\quad M[i][j] = \min(m_1, m_2, m_3) \\
\text{return } M[i][j]
\]
Matrix and DAG of Computation

Matrix M:

Figure: Dependency of matrix entries in the recursive algorithm of previous slide
Removing Recursion to obtain Iterative Algorithm

EDIST(A[1..m], B[1..n])

int M[0..m][0..n]

for i = 1 to m do M[i,0] = i
for j = 1 to n do M[0,j] = j

for i = 1 to m do
  for j = 1 to n do
    M[i][j] = min \[\begin{align*}
      & [x_i = y_j] + M[i-1][j-1], \\
      & 1 + M[i-1][j], \\
      & 1 + M[i][j-1]
    \end{align*}\]

Analysis

Running time is $O(mn)$. 
Removing Recursion to obtain Iterative Algorithm

$EDIST(A[1..m], B[1..n])$

```plaintext
int M[0..m][0..n]
for $i = 1$ to $m$ do $M[i, 0] = i$
for $j = 1$ to $n$ do $M[0, j] = j$
for $i = 1$ to $m$ do
    for $j = 1$ to $n$ do
        $M[i][j] = \min \begin{cases} 
            [x_i = y_j] + M[i - 1][j - 1], \\
            1 + M[i - 1][j], \\
            1 + M[i][j - 1] 
        \end{cases}$
```

Analysis

Running time is $O(mn)$. 
EDIST(A[1..m], B[1..n])

int M[0..m][0..n]

for $i = 1$ to $m$ do $M[i, 0] = i$

for $j = 1$ to $n$ do $M[0, j] = j$

for $i = 1$ to $m$ do
  for $j = 1$ to $n$ do
    $M[i][j] = \min\left\{\begin{array}{l}
[x_i = y_j] + M[i - 1][j - 1],
1 + M[i - 1][j],
1 + M[i][j - 1]
\end{array}\right\}$

Analysis

1. Running time is $O(mn)$.
2. Space used is $O(mn)$. 
**Matrix and DAG of Computation (revisited)**

Matrix $M$:

![Matrix M Diagram](image)

**Figure:** Iterative algorithm in previous slide computes values in row order.
The DP algorithm finds the minimum edit distance in $O(nm)$ space and time.

**Question:** Can we find a specific alignment which achieves the minimum?
Finding an Optimum Solution

The DP algorithm finds the minimum edit distance in \( O(nm) \) space and time.

**Question:** Can we find a specific alignment which achieves the minimum?

**Exercise:** Show that one can find an optimum solution after computing the optimum value. Key idea is to store back pointers when computing \( \text{Opt}(i, j) \) to know how we calculated it. See notes for more details.
Dynamic Programming Template

1. Come up with a recursive algorithm to solve problem
2. Understand the structure/number of the subproblems generated by recursion
3. Memoize the recursion → DP
   - set up compact notation for subproblems
   - set up a data structure for storing subproblems
4. Iterative algorithm
   - Understand dependency graph on subproblems
   - Pick an evaluation order (any topological sort of the dependency dag)
5. Analyze time and space
6. Optimize
Part III

Knapsack
Knapsack Problem

Input  Given a Knapsack of capacity $W$ lbs. and $n$ objects with $i$th object having weight $w_i$ and value $v_i$; assume $W$, $w_i$, $v_i$ are all positive integers

Goal  Fill the Knapsack without exceeding weight limit while maximizing value.
Knapsack Problem

Input  Given a Knapsack of capacity $W$ lbs. and $n$ objects with $i$th object having weight $w_i$ and value $v_i$; assume $W$, $w_i$, $v_i$ are all positive integers

Goal  Fill the Knapsack without exceeding weight limit while maximizing value.

Basic problem that arises in many applications as a sub-problem.
## Knapsack Example

### Example

<table>
<thead>
<tr>
<th>Item</th>
<th>$l_1$</th>
<th>$l_2$</th>
<th>$l_3$</th>
<th>$l_4$</th>
<th>$l_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>1</td>
<td>6</td>
<td>18</td>
<td>22</td>
<td>28</td>
</tr>
<tr>
<td>Weight</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

If $W = 11$, the best is $\{l_3, l_4\}$ giving value 40.
Knapsack Example

Example

<table>
<thead>
<tr>
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<tr>
<td>Weight</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

If $W = 11$, the best is $\{l_3, l_4\}$ giving value 40.

Special Case

When $v_i = w_i$, the Knapsack problem is called the **Subset Sum Problem**.
For the following instance of Knapsack:

<table>
<thead>
<tr>
<th>Item</th>
<th>$l_1$</th>
<th>$l_2$</th>
<th>$l_3$</th>
<th>$l_4$</th>
<th>$l_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>1</td>
<td>6</td>
<td>16</td>
<td>22</td>
<td>28</td>
</tr>
<tr>
<td>Weight</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

and weight limit $W = 15$. The best solution has value:

(A) 22
(B) 28
(C) 38
(D) 50
(E) 56
Greedy Approach

1. Pick objects with greatest value

   Let \( W = 2, w_1 = w_2 = 1, w_3 = 2, v_1 = v_2 = 2 \) and \( v_3 = 3 \);

2. Pick objects with smallest weight

   Let \( W = 2, w_1 = 1, w_2 = 2, v_1 = 1 \) and \( v_2 = 3 \);

3. Pick objects with largest \( \frac{v_i}{w_i} \) ratio

   Let \( W = 4, w_1 = w_2 = 2, w_3 = 3, v_1 = v_2 = 3 \) and \( v_3 = 5 \);

Aside: Can show that a slight modification always gives half the optimum profit: pick the better of the output of this algorithm and the largest value item. Also, the algorithms gives better approximations when all item weights are small when compared to \( W \).
Greedy Approach

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   - Let $W = 2$, $w_1 = w_2 = 1$, $w_3 = 2$, $v_1 = v_2 = 2$ and $v_3 = 3$; greedy strategy will pick $\{3\}$, but the optimal is $\{1, 2\}$

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4. Aside: Can show that a slight modification always gives half the optimum profit: pick the better of the output of this algorithm and the largest value item. Also, the algorithms gives better approximations when all item weights are small when compared to \( W \).
First guess: $\text{Opt}(i)$ is the optimum solution value for items $1, \ldots, i$.

**Observation**

Consider an optimal solution $O$ for $1, \ldots, i$

Case item $i \notin O$ $O$ is an optimal solution to items $1$ to $i - 1$

Case item $i \in O$ Then $O - \{i\}$ is an optimum solution for items $1$ to $i - 1$ in knapsack of capacity $W - w_i$. 
Towards a Recursive Algorithms

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Consider an optimal solution $\mathcal{O}$ for $1, \ldots, i$

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*Subproblems depend also on remaining capacity. Cannot write subproblem only in terms of $\text{Opt}(1), \ldots, \text{Opt}(i - 1)$.***
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$\text{Opt}(i, w)$: optimum profit for items $1$ to $i$ in knapsack of size $w$

Goal: compute $\text{Opt}(n, W)$
Dynamic Programming Solution

**Definition**

Let $\text{Opt}(i, w)$ be the optimal way of picking items from $1$ to $i$, with total weight not exceeding $w$.

$$
\text{Opt}(i, w) = \begin{cases} 
0 & \text{if } i = 0 \\
\text{Opt}(i - 1, w) & \text{if } w_i > w \\
\max \left\{ \text{Opt}(i - 1, w), \text{Opt}(i - 1, w - w_i) + v_i \right\} & \text{otherwise}
\end{cases}
$$

Number of subproblem generated by $\text{Opt}(n, W)$ is $O(nW)$. 
An Iterative Algorithm

```
for \( w = 0 \) to \( W \) do
    \( M[0, w] = 0 \)
for \( i = 1 \) to \( n \) do
    for \( w = 1 \) to \( W \) do
        if \( (w_i > w) \) then
            \( M[i, w] = M[i - 1, w] \)
        else
            \( M[i, w] = \max(M[i - 1, w], M[i - 1, w - w_i] + v_i) \)
```

Running Time

Running time is \( O(nW) \); input has size \( O(n + \log W + \sum_{i=1}^{n} (\log v_i + \log w_i)) \); so running time not polynomial but "pseudo-polynomial!"
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\[
\begin{align*}
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Introducing a Variable

For the Knapsack problem obtaining a recursive algorithm required introducing a new variable, namely the size of the knapsack.

This is a key idea that recurs in many dynamic programming problems.
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How do we figure out when this is possible?

Heuristic answer that works for many problems: Try divide and conquer or obvious recursion: if problem is not decomposable then introduce the “information” required to decompose as new variable(s). Will see several examples to make this idea concrete.
Knapsack Algorithm and Polynomial time

1. Input size for Knapsack:
   \[ O(n) + \log W + \sum_{i=1}^{n}(\log w_i + \log v_i). \]

2. Running time of dynamic programming algorithm: \( O(nW) \).

3. Not a polynomial time algorithm.

Example: \( W = 2n \) and \( w_i, v_i \in [1..2n] \). Input size is \( O(n^2) \), running time is \( O(n^2n) \) arithmetic/comparisons.

Algorithm is called a pseudo-polynomial time algorithm because running time is polynomial if numbers in input are of size polynomial in the combinatorial size of problem.

Knapsack is NP-Hard if numbers are not polynomial in \( n \).
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