Given on exam:
- Prob. inequalities
- NP-hard problems
- Standard rubrics — DP, Graph reduction, NP-hardness...

"Prove" → we want a proof
"Prove" → we do not want a proof
Linear arrangement problem

Input: Directed graph $G = (V, E)$
Output: Indexing of $V = \{v_1, v_2, \ldots, v_n\}$
  s.t. $\#$ edges $v_i \rightarrow v_j$ with $i < j$
is maximized.

If $G$ is a dag? Topological sort!
In general NP-hard

Question: Design a fast 2-approximation algorithm.
We know $\text{OPT} \leq E$.

1. Sort by outdegree?

2. Pick arbitrary ordering.
   
   If $\geq E/2$ forward edges, done.
   
   Else reverse everything!
SP 2015 Final #2

$m$ soldiers
$n$ tasks
$L \geq k$ soldiers qualified for each task

Select a set $S$ of soldiers maximizing $\#$ tasks with ONE qualified soldier in $S$.

(a) Choose each soldier with prob $p$.

$E[\#tasks] = \sum_{i=1}^{n} \Pr(\text{task } i \text{ is completed})$

$= n \cdot p \cdot (1-p)^{k-1} \cdot k$

(b) Best value of $p =$?
\[ \frac{d}{dp} p(1-p)^{k-1} = (1-p)^{k-1} + p(k-1)(1-p)^{k-2} = 0 \]

\[ (1-p) = p(k-1) \]

\[ 1 = pk - p + p \]

\[ p = \frac{1}{k} \]

\[ \Rightarrow E[\text{#tasks}] = n \cdot (1 - \frac{1}{k})^{k-1} \approx n/e \]

\[ \Rightarrow E[\text{approx}] \approx 1/e \checkmark \]
$\mathcal{F} = O(|E| \cdot |F^*|)$ time

Orlin: $O(|V|E)$ time

- edges have capacities and/or have lower bounds
- vertices have capacities and/or have lower bounds on incoming flow (or outgoing flow)
- multiple sources, multiple sinks, or no Feasible? max. value
- vertices can have non-zero balances
- Flow decomposition - integer flow
- edge-disjoint paths
- vertex-disjoint paths
- max. bipartite matching
- disjoint path covers of dags
- path covers of dags
- assignment/tuple selection
n x n checkerboard with some squares removed

Cover every square exactly once with dominoes: 2x1 or 1x2 rectangles

Bipartite matching \((L \cup R, E)\)

- \(L\) = white squares
- \(R\) = black squares
- \(E\) = adj squares - share boundary side

Cover with dominoes

Perfect matching time: \(O(VE)\)

\(= O(n^4)\)
Double Hamiltonian circuit is NP-hard

Reduce from Ham cycle

Given $G(V, E)$ construct $G' = (V', E')$

Attach a lollipop to every vertex

$G$ has Ham cycle $\rightarrow G'$ has double Ham. $\checkmark$

$G'$ has double Ham.
case analysis:
within each gadget
\[ 2 \text{Ham} \rightarrow x \rightarrow y \rightarrow z \rightarrow x \rightarrow y \rightarrow x \rightarrow z \rightarrow y \rightarrow x \rightarrow y \]
\[ \approx \log n \]

Delete gadgets, left Ham cycle in \( G \).

poly time \( \checkmark \)
A triple Hamiltonian cycle = closed walk that visits every vertex exactly 3 times.
Max flow notes problem 6.

\[ G = (V, E) \text{ Flow network} \]
\[ \text{every edge has capacity 1} \]
\[ \text{shortest path from s to t is } \geq d. \]
\[ \Rightarrow d \]

(a) \[ \max \text{ flow } \leq \frac{E}{d} \]

Suppose \( f^* \) is max flow.

Decompose into \( |f^*| \) paths edge-disjoint.

Each path has length \( \geq d \).

Total # edges covered by flow \( \geq d \cdot |f^*| \)

\[ |f^*| \leq \frac{E}{d} \]
Nuts and Bolts notes Problem 7

Diagram:

```
10 - 1 - 3 - 4 - 11 - 8 - 12
```

Matrix:

```
  1 2 3 4 5 6 7
X = 3 1 4 7 2 1 2 0 8
next 3 6 7 1 2 4 5
```

Given: X randomly permuted
X[next[i]] is successor of X[i] in sorted order

Given x, is x in X?

Goal: O(\(\sqrt{n}\)) time

Algorithm:

Choose k elements of X at random.
Find largest sample smaller than x. \(- O(k) time\)
Scan forward to \(\geq x\) \(- O(n/k) \text{exp. time}\)
\(2n/(k+1)\)
Spring 2015 Final #6

Given linear inequalities

\[ a_i x + b_i y \leq c_i \]

Describe LP whose solution describes largest square in feasible polygon P.
Describe □ by center (x, y)
edge length l

\[
\begin{align*}
\max l \\
\text{s.t. } a_i(x + \frac{b}{2}) + b_i(y + \frac{b}{2}) & \leq c_i \text{ for all } i \\
+ & + \\
+ & - \\
- & - \\
\end{align*}
\]

4 constraints for each i
$O(n \log^5 n)$
Assign random priority to every vertex.

\[ S = \{ v \mid \text{priority}(v) \geq \min_{uv} \text{priority}(u) \} \]

\[ \text{Pr}[S \text{ is a vertex cover}] = ? \]

\[ \text{pri}(u) \leq \text{pri}(v) \]
(b) \( G \) = cycle of length \( n \)

\[
E[|S|] = \sum_{i=1}^{\frac{n}{2}} \Pr[v_i \in S] = \frac{2}{3} \left( 1 - \frac{1}{n} \right)
\]

\[
= \left( \frac{2n}{3} \right)
\]

(c) \( G \) = star with \( n-1 \) leaves

\[
E[|S|] = \sum_{i=1}^{\frac{n}{2}} \Pr[v_i \in S]
\]

\[
= (n-1) \Pr[\text{leaf} \in S] + \Pr[\text{center} \in S]
\]

\[
= \frac{n-1}{2} + \frac{n-1}{n} = \left( \frac{2n + \frac{1}{2} - \frac{1}{n}}{n} \right)
\]
(b) star. Choose $S_1, S_2, \ldots, S_N$ independently

How large $N$ so that some $S_i$ is \( \min \text{ vertex cover?} \)

\[
\Pr[S = \min \text{ VC}] = \Pr[priority(v_n) \geq priority(v_i) \text{ for all } i]
\]

\[
= \frac{1}{n}
\]

\[
\Pr[\min \text{ VC is one of } S_1 \ldots S_N] =
\]

\[
= 1 - \Pr[\text{none of } S_1 \ldots S_N \text{ is } \min \text{ VC}]
\]

\[
= 1 - (1 - \frac{1}{n})^N \geq 1 - \frac{1}{n^a}
\]

If $N \geq an / \ln n$

\[
= \sqrt{2(n \log n)}
\]
\[ N \geq \alpha n \log n \]
\[
\frac{N}{n} \geq \alpha \log n
\]
\[ e^{N/n} \geq \alpha n \]
\[ e^{-N/n} \approx \left(1 - \frac{1}{n}\right)^N \leq \frac{1}{n^\alpha} \]
\[ 1 - \left(1 - \frac{1}{n}\right)^N \geq 1 - \frac{1}{n^\alpha} \]
Spring 2016 Final #3

n boxes, \( h, w, d \) in cm

\( 10 \leq h, w, d \leq 20 \)

Nest boxes so that \( \# \) visible boxes is as small as possible
Solution 1

Disjoint path cover

Define dag $G = (V, E)$

$V =$ boxes

$E = \{ u \rightarrow v \mid v \text{ fits inside } u \text{ after some rotation} \}$

$\min \dim u < \min \dim v$

Path = seq of nested boxes

Outermost is only visible vertex

$\min \# \text{ visible boxes} \iff \min \# \text{ disjoint paths cover } G$

Time = $O(VE) = O(n^3)$ time
Solution 2: Matching

\[ G = (V_L \cup V_R, E) \]

\[ V_L = \text{boxes} \]

\[ V_R = \text{boxes} \]

\[ E = \{ (u_L, v_R) \mid u \text{ fits inside } v_R^3 \} \]

Intuition: \( uv \in M \) if \( v \) is the smallest box containing \( u \)

Visible box = unmatched vertex in \( L \)

\[ O(VE) = O(n^3) \text{ time} \]