Given on exam:

Prob. inequalities
NP-hard problems
Standard rubrics — DP, Graph reduction, NP-hardness...

"Prove" → we want a proof

~~"Prove"~~ → we do not want a proof

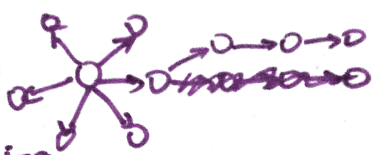
Linear arrangement problem

Input: Directed graph $G=(V,E)$
Output: Indexing of $V=\{v_1, v_2, \dots, v_n\}$ s.t. # edges $v_i \to v_j$ with $i<j$ is maximized.

If G is a dag? Topological sort!
In general NP-hard
Question: Design a fast 2-approx algorithm.

$\boxed{\text{We know } OPT \le E.}$

① ~~Sort by outdegree?~~



② Pick arbitrary ordering.
If $\ge E/2$ forward edges, done.
Else reverse everything!

SP 2015 Final #2

m soldiers
n tasks
↳ k soldiers qualified for each task

Select a set S of soldiers maximizing # tasks with ONE qualified ~~select~~ soldier in S.

ⓐ Choose each soldier with prob p.

$$E[\#\text{tasks}] = \sum_{i=1}^{n} \Pr(\text{task } i \text{ is completed})$$

$$= \boxed{n \cdot p \cdot (1-p)^{k-1} \cdot k}$$

ⓑ Best value of p = ?

$$\frac{d}{dp} p(1-p)^{k-1} = (1-p)^{k-1} - p(k-1)(1-p)^{k-2} = 0$$

$$(1-p) = p(k-1)$$

$$1 = pk - p + p$$

$$\boxed{p = 1/k}$$

$$\cancel{p = (1-p)^{k-1}}$$

$\hookrightarrow E[\#\text{tasks}]$

$$= n \cdot \underbrace{\left(1 - \frac{1}{k}\right)^{k-1}}_{\approx n/e}$$

(c) $O(1)$-approx algo

$E[\text{approx}] \approx 1/e$ ✓

FF: $O(E \cdot |F^*|)$ time

Orlin: $O(VE)$ time

- edges have capacities and/or have lower bounds
- vertices have capacities and/or have lower bounds on incoming flow (or outgoing flow)
- multiple sources, multiple sinks, or no / or no
  - feasible?
  - max. value
- vertices can have non-zero balances
- Flow decomposition — integer flow

- edge-disjoint paths
- vertex-disjoint paths
- max. bipartite matching
- disjoint path covers of dags
- ——— path covers of dags
- assignment/tuple selection

$n \times n$ checkerboard with some squares removed

Cover every square exactly once with dominoes: 2×1 or 1×2 rectangles

Bipartite matching $(L \cup R, E)$

L = white squares R = black squares

E = adj squares — share boundary side

domino = edge

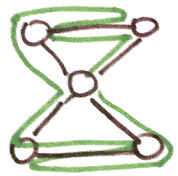
Cover with dominos $\leftrightarrow$ perfect matching

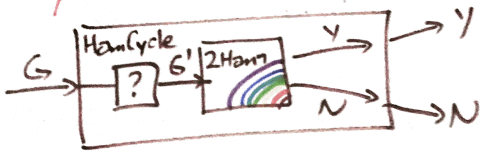
time: $O(VE)$

$= \boxed{O(n^4)}$

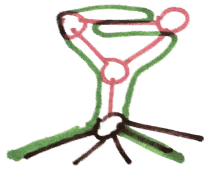
Double Hamiltonian circuit is NP-hard

Reduce from Ham cycle





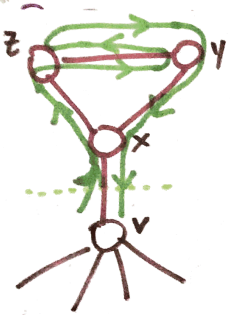
Given $G(V,E)$ construct $G'=(V',E')$



Attach a lollipop to every vertex

G has Ham cycle $\rightarrow$ G' has double Ham. ✓

G' has double Ham.



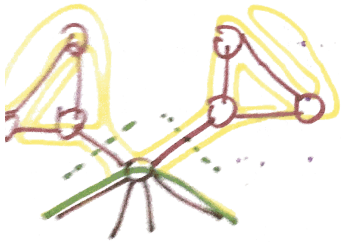
case analysis:
within each gadget
2 Ham $v \to x \to z \to y \to z \to y \to x \to v$
wlog

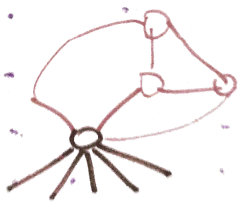
Delete gadgets, left Ham cycle in G.

poly time ✓

Sp 2016 Final #1

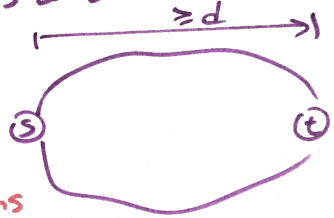
A triple Hamiltonian cycle = closed walk that visits every vertex exactly 3 times.





Max Flow notes problem 6.

$G=(V,E)$ Flow network
every edge has capacity 1
shortest path from s to t is $\geq d$.

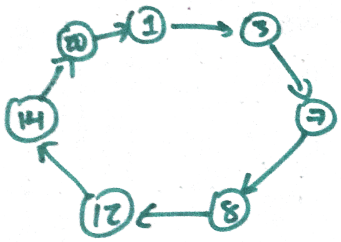


(a) max Flow $\leq E/d$

Spose f^* is max flow
Decompose into $|f^*|$ paths edge-disjoint
each path has length $\geq d$
Total #edges covered by flow $\geq d \cdot |f^*|$
$\leq E$

$|f^*| \leq \frac{E}{d}$ □

Nuts+Bolts notes Problem 7



	1	2	3	4	5	6	7
X	3	14	7	1	12	20	8
next	3	6	7	1	2	4	5

Given X randomly permuted

X[next[i]] is successor of X[i] in sorted order

Given x, is x in X?

9? Goal: $O(\sqrt{n})$ time

Algo:

Choose k elements of X at random.

Find largest sample smaller than x. — $O(k)$ time

Scan forward to $\geq x$ — $O(n/k)$ exp. time.

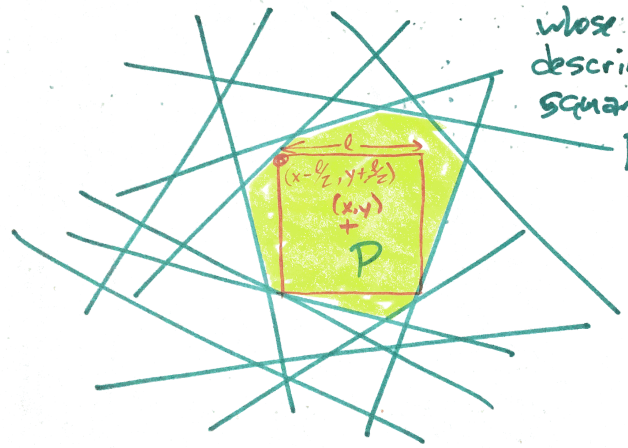
$2n/(k+1)$

Spring 2015 Final #6

Given linear inequalities

$a_i x + b_i y \leq c_i$

Describe LP whose solution describes largest square in feasible polygon P



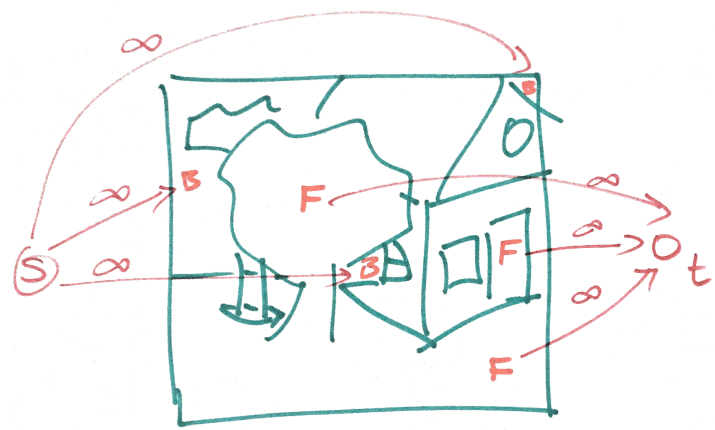
Describe □ by center (x,y)
edge length ℓ

$$\max \ell$$

4 constraints for each i

$$\text{s.t. } a_i(x \pm \ell/2) + b_i(y \pm \ell/2) \leq c_i \quad \text{for all } i$$

$$\begin{matrix} + & + \\ + & - \\ - & - \end{matrix}$$

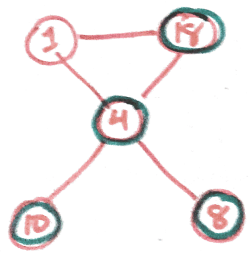


$O(n \log^5 n)$

Sp 2016 Final #5

Assign random priority (real number between 0 and 1) to every vertex

$S = \{v \mid \text{priority}(v) \geq \min_{uv} \text{priority}(u)\}$

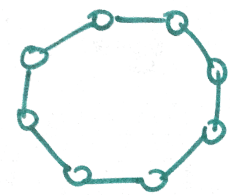


ⓐ $Pr[S \text{ is a vertex cover}] = ?$

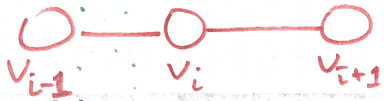
$\boxed{1}$

$\text{pri}(u) \leq \text{pri}(v)$

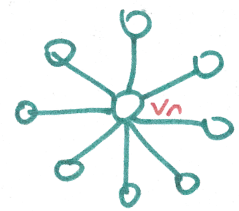
ⓑ G = cycle of length n



$E[|S|] = \sum_{i=1}^{n} Pr[v_i \in S] = \sum_{i=1}^{n} \frac{2}{3} = \boxed{\frac{2n}{3}}$



ⓒ G star with n-1 leaves



$E[|S|] = \sum_{i=1}^{n} Pr[v_i \in S]$

$= (n-1) Pr[\text{leaf} \in S] + Pr[\text{center} \in S]$

$= \frac{n-1}{2} + \frac{n-1}{n} = \boxed{\frac{n}{2} + \frac{1}{2} - \frac{1}{n}}$

ⓓ star. Choose $S_1, S_2, \ldots S_N$ independently

How large N so that some S_i is min vertex cover? whp

$Pr[S = \min VC] = Pr[\text{priority}(v_n) \geq \text{priority}(v_i) \text{ for all } i]$

$= 1/n$

$Pr[\min VC \text{ is one of } S_1 \ldots S_N] =$

$= 1 - Pr[\text{none of } S_1 \ldots S_N \text{ is min VC}]$

$= 1 - (1 - \frac{1}{n})^N \geq 1 - \frac{1}{n^\alpha}$

if $N \geq \alpha n \ln n$

$= \Omega(n \log n)$

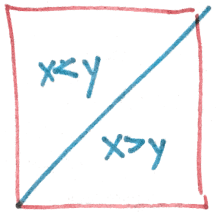
$\boxed{N \geq \boxed{\alpha n \log n}}$

$N/n \geq \alpha \log n$

$e^{N/n} \geq n^\alpha$

$e^{-N/n} \approx (1 - \frac{1}{n})^N \leq \frac{1}{n^\alpha}$

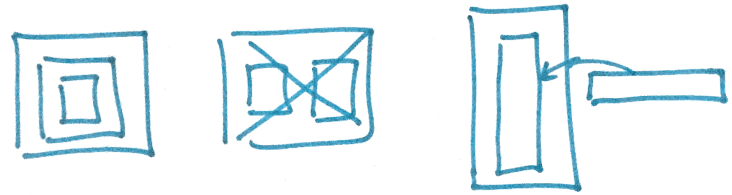
$1 - (1 - \frac{1}{n})^N \geq 1 - \frac{1}{n^\alpha}$



Spring 2016 Final #3

n boxes, h, w, d in cm

$10 \le h, w, d \le 20$



Nest boxes so that # visible boxes is as small as possibe

Solution 1

Disjoint path cover

Define dag $G=(V,E)$

V = boxes

$E = \{u \to v \mid v$ fits inside u after some rotation$\}$ dag ✓

min dim v < min dim u

path = seq of nested boxes

outermost is only visible

min # visible boxes $\rightleftarrows$ min # vertex disjoint paths cover G

time $= O(VE) = O(n^3)$ time

Solution 2: Matching

$G = (L \cup R, E)$

L = boxes

R = boxes

$E = \{u_L v_R \mid u$ fits inside $v\}$

max matching M in G

Intuition $uv \in M$ if v is smallest box containing u

$\boxed{\text{visible box} = \text{unmatched vertex in } L}$

$O(VE) = O(n^3)$ time