Given on exam:
Prob. inequalities
NP-hard problems
Standard rubrics — DP, Graph reduction, NP-hardness...

"Prove" → we want a proof
"Prove" → we do not want a proof

Linear arrangement problem
Input: Directed graph $G = (V, E)$
Output: Indexing of $V = \{v_1, v_2, ..., v_n\}$
s.t. #edges $v_i \rightarrow v_j$ with $i < j$
is maximized.

If $G$ is a dag? Topological sort!
In general NP-hard
Question: Design a fast $2$-approximation algorithm.

We know $OPT \leq E$.

1. Sort by outdegree.
2. Pick arbitrary ordering.
   If $\geq E/2$ forward edges, done.
   Else reverse everything.

SP 2015 Final #2
$m$ soldiers
$n$ tasks
$\geq k$ soldiers qualified for each task
Select a set $S$ of soldiers
maximizing # tasks with ONE qualified soldier in $S$.

(a) Choose each soldier with prob $p$.
$$E[\#\text{tasks}] = \sum_{i=1}^{n} \Pr(\text{task } i \text{ is completed})$$
$$= n \cdot p \cdot (1-p)^{k-1} \cdot k$$
(b) Best value of $p =$?
\[ \frac{d}{dp} p(1-p)^k = (1-p)^{k-1} + p(k-1)(1-p)^{k-2} = 0 \]

\[ (1-p) = p(k-1) \]

\[ 1 = pk - p + p \]

\[ P = \left(1 - \frac{p}{k}\right)^k \]

\[ E(\text{#tasks}) = n \cdot (1 - \frac{1}{k}) \]

\[ E(\text{approx}) \approx \frac{n}{e} \]

\[ 0(1) \text{- approx algo} \]

\[ \Box \]

FF: \( O(E |F|) \) time

Orlin: \( O(VE) \) time

- edges have capacities and/or have lower bounds
- vertices have capacities and/or have lower bounds
  - on incoming flow (or outgoing flow)
  - multiple sources, multiple sinks, or no
  - feasible? max. value
- vertices can have non-zero balances
- Flow decomposition - integer flow

\[ n \times n \text{ checkerboard with some squares removed} \]

\[ \text{cover every square exactly once with dominoes: } 2 \times 1 \text{ or } 1 \times 2 \text{ rectangles} \]

Bipartite matching (L, R, E)

\[ L = \text{white squares} \quad R = \text{black squares} \]

\[ E = \text{adj squares - share boundary side} \]

\[ \text{domino} = \text{edge} \]

\[ \text{Cover with domino perfect matching} \quad \text{time: } O(VE) = O(n^2) \]
Double Hamiltonian circuit is NP-hard
Reduce from Ham cycle

\[ \text{HamCycle} \rightarrow \text{2Ham} \]

Given \( G(V,E) \) construct \( G'(V',E') \)
Attach a lollipop to every vertex

\( G \) has Ham cycle \( \rightarrow \) \( G' \) has double Ham. √
\( G' \) has double Ham.

Case analysis:
within each gadget
\[ \text{2Ham} \quad \frac{\text{poly time}}{\log n} \]
Delete gadgets, left Ham cycle in \( G \).

Sp 2016 Final #1
A triple Hamiltonian cycle = closed walk that visits every vertex exactly 3 times.

Max flow notes problem 6.
\( G=(V,E) \) Flow network
every edge has capacity 1
shortest path from \( s \) to \( t \) is \( \geq d \).

3) max flow \( \leq \frac{E}{d} \)

Suppose \( f^* \) is max flow
Decompose into \( |f^*| \) paths edge-disjoint
each path has length \( \geq d \)
Total #edges covered by flow \( \geq d \cdot |f^*| \)
\[ \frac{|f^*| \leq E}{\square} \]
Nuts + Bolts notes Problem 7

Given X randomly permuted
X[\text{next}[i]] is successor of \text{[i]} in sorted order

Given x, is x in X?

Goal: \( O(\sqrt{n}) \) time

Algorithm:
- Choose \( k \) elements of \( X \) at random.
- Find largest sample smaller than \( x \). - \( O(k) \) time
- Scan forward to \( \geq x \) - \( O(n/k) \) exp. time.
- \( 2n/k + 1 \)

Describe \( \square \) by center \((x,y)\)
edge length \( l \)

\[
\max l \quad \text{s.t. } a_i(x + \frac{l}{2}) + b_i(y + \frac{l}{2}) \leq c_i \quad \text{for all } i
\]

4 constraints for each \( i \)

Spring 2015 Final #6

Given linear inequalities
\( a_i x + b_i y \leq c_i \)

Describe LP whose solution describes largest square in feasible polygon \( P \)

Describe \( O(n \log^4 n) \)
Sp 2016 Final #5

a. Assign random priority to every vertex $S = \{ v \mid \text{priority}(v) \geq \min_{uv} \text{priority}(u) \}$

Pr[$S$ is a vertex cover] = 1

b. $G$ cycle of length $n$

$$E[|S|] = \sum_{i=1}^{\frac{n}{2}} \Pr[v_i \in S] = \frac{n}{2} \cdot \frac{2}{3} = \frac{2n}{3}$$

c. $G$ star with $n-1$ leaves

$$E[|S|] = \sum_{i=1}^{\frac{n}{2}} \Pr[v_i \in S] = (n-1)\Pr[\text{leaf} \in S] + \Pr[\text{center} \in S]$$

$$= \frac{n-1}{2} + \frac{n-1}{n} = \frac{n+1-1}{2} = \frac{n}{2}$$

d. Star. Choose $S_1, S_2, \ldots, S_n$ independently

How large $N$ so that some $S_i$ is a min vertex cover $\text{w.h.p.}$

$$\Pr[S = \text{min VC}] = \Pr[\text{priority}(v_i) \geq \text{priority}(u_i) \text{ for all } i]$$

$$= \frac{1}{n}$$

$$\Pr[\text{no min VC is one of } S_1, S_2, \ldots, S_n]$$

$$= 1 - \Pr[\text{none of } S_1, S_2, \ldots, S_n \text{ is min VC}]$$

$$= 1 - (1 - \frac{1}{n})^N \geq 1 - \frac{1}{n^a}$$

if $N \geq a \log n$

$$= 2(n \log n)$$

$$N \geq \alpha n \log n$$

$$\frac{N}{n} \geq \alpha \log n$$

$$e^{N/n} \approx n^a$$

$$e^{-N/n} \approx (1 - \frac{1}{n})^N \leq \frac{1}{n^a}$$

$$1 - (1 - \frac{1}{n})^N \geq 1 - \frac{1}{n^a}$$
Solution 1
Disjoint path cover
Define dag \( G = (V, E) \)
\( V = \) boxes
\( E = \{ u \to v \mid v \text{ fits inside } u \} \) after some rotation \( \S \) dag \( \S \)
\( \text{min dim } u < \text{min dim } v \)
path = seq of nested boxes
outermost is only visible vertex
\( \text{min # visible boxes} \leq \text{min # disjoint paths cover } G \)

time = \( O(VE) = O(n^3) \) time

Solution 2: Matching
\( G = (L, U, R, E) \)
\( L = \) boxes
\( U = \) boxes
\( R = \) boxes
\( E = \{ u \to v \mid u \text{ fits inside } v \} \)
Intuition \( u \in \text{match} \) \( \text{if } v \text{ is smallest box containing } u \)
Visible box = unmatched vertex in \( L \)
\( O(VE) = O(n^3) \) time