

Well, ya turn left by the fire station in the village and take the old post road by the reservoir and... no, that won't do.

Best to continue straight on by the tar road until you reach the schoolhouse and then turn left on the road to Bennett's Lake until... no, that won't work either.

East Millinocket, ya say? Come to think of it, you can't get there from here.

— Robert Bryan and Marshall Dodge,
Bert and I and Other Stories from Down East (1961)

Hey farmer! Where does this road go?

Been livin' here all my life, it ain't gone nowhere yet.

Hey farmer! How do you get to Little Rock?

Listen stranger, you can't get there from here.

Hey farmer! You don't know very much do you?

No, but I ain't lost.

— Michelle Shocked, "Arkansas Traveler" (1992)

CHAPTER 8

Shortest Paths

This is the Spring 2016 revision in the new skin; it still needs significant revision. In particular, several figures need to be redrawn in OmniGraffle.



8.1 Introduction

Suppose we are given a weighted *directed* graph $G = (V, E, w)$ with two special vertices, and we want to find the shortest path from a *source* vertex s to a *target* vertex t . That is, we want to find the directed path p starting at s and ending at t that minimizes the function

$$w(p) := \sum_{u \rightarrow v \in p} w(u \rightarrow v).$$

For example, if I want to answer the question “What’s the fastest way to drive from my old apartment in Champaign, Illinois to my wife’s old apartment in Columbus, Ohio?”, I might use a graph whose vertices are cities, edges are roads, weights are driving times,

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s is Champaign, and t is Columbus.¹ The graph is directed, because driving times along the same road might be different in different directions. (At one time, there was a speed trap on I-70 just east of the Indiana/Ohio border, but only for eastbound traffic.)

Perhaps counter to intuition, we will allow the weights on the edges to be negative. Negative edges make our lives complicated, because the presence of a negative cycle might imply that there is no shortest path. In general, a shortest path from s to t exists if and only if there is *at least one* path from s to t , but there is no path from s to t that touches a negative cycle. If any negative cycle is reachable from s and can reach t , we can always find a shorter path by going around the cycle one more time.



I'm abusing terminology here; by definition, a path cannot repeat vertices. Negative cycles mean that there may be no shortest **walk** from s to t . As long as t is reachable from s , there is always a shortest **path**; it's just NP-hard to compute. On the other hand, if there is a shortest **walk** from s to t , that walk is in fact a path, and therefore the shortest path, from s to t . Blerg.

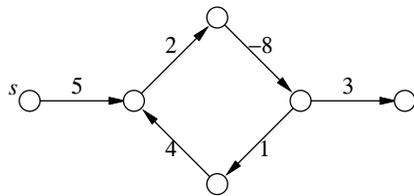


Figure 8.1. There is no shortest path from s to t .

Almost every algorithm known for solving this problem actually solves (large portions of) the following more general *single source shortest path* or *SSSP* problem: Find the shortest path from the source vertex s to *every* other vertex in the graph. This problem is usually solved by finding a *shortest path tree* rooted at s that contains all the desired shortest paths.

It's not hard to see that if shortest paths are unique, then they form a tree, because any subpath of a shortest path is itself a shortest path. If there are multiple shortest paths to some vertices, we can always choose one shortest path to each vertex so that the union of the paths is a tree. If there are shortest paths to two vertices u and v that diverge, then meet, then diverge again, we can modify one of the paths without changing its length so that the two paths only diverge once.

Although they are both optimal spanning trees, shortest-path trees and minimum spanning trees are very different creatures. Shortest-path trees are rooted and directed; minimum spanning trees are unrooted and undirected. Shortest-path trees are most naturally defined for directed graphs; only undirected graphs have minimum spanning trees. If edge weights are distinct, there is only one minimum spanning tree, but every

¹West on Church, north on Prospect, east on I-74, south on I-465, east on Airport Expressway, north on I-65, east on I-70, north on Grandview, east on 5th, north on Olentangy River, east on Dodridge, north on High, west on Kelso, south on Neil. Depending on traffic. We both live in Urbana now.

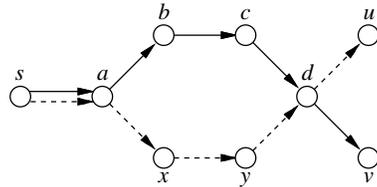


Figure 8.2. If $s \rightarrow a \rightarrow b \rightarrow c \rightarrow d \rightarrow v$ and $s \rightarrow a \rightarrow x \rightarrow y \rightarrow d \rightarrow u$ are shortest paths, then $s \rightarrow a \rightarrow b \rightarrow c \rightarrow d \rightarrow u$ is also a shortest path.

source vertex induces a different shortest-path tree; moreover, it is possible for every shortest path tree to use a different set of edges from the minimum spanning tree.

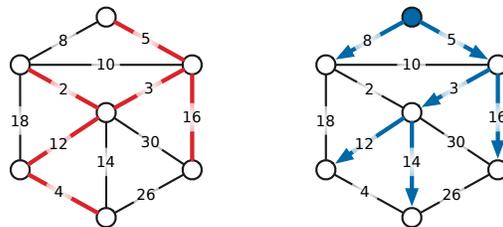
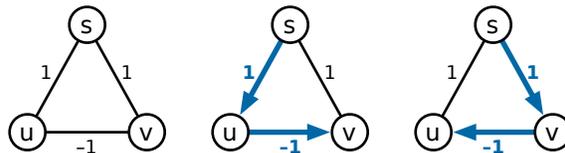


Figure 8.3. A minimum spanning tree and a shortest path tree (rooted at the top vertex) of the same undirected graph.

8.2 Warning!

Throughout this chapter, we will explicitly consider *only* directed graphs. All of the algorithms described in this lecture also work for undirected graphs with some minor modifications, *but only if negative edges are prohibited*. Dealing with negative edges in undirected graphs is considerably more subtle. We cannot simply replace every undirected edge with a pair of directed edges, because this would transform any negative edge into a short negative cycle. Subpaths of an *undirected* shortest path that contains a negative edge are *not* necessarily shortest paths; consequently, the set of all undirected shortest paths from a single source vertex may not define a tree, even if shortest paths are unique.



An undirected graph where shortest paths from s are unique but do not define a tree.

A complete treatment of undirected graphs with negative edges is beyond the scope of this chapter, or even the entire book. I will only mention, for people who want to

follow up via Google, that a *single* shortest path in an undirected graph with negative edges can be computed in $O(VE + V^2 \log V)$ time, by a reduction to maximum weighted matching.

8.3 The Only SSSP Algorithm

Just like graph traversal and minimum spanning trees, there are several different SSSP algorithms, but they are all special cases of the a single generic algorithm, first proposed by Lester Ford in 1956, and independently by George Dantzig in 1957.² Each vertex v in the graph stores two values, which (inductively) describe a *tentative* shortest path from s to v .

- $dist(v)$ is the length of the tentative shortest $s \rightsquigarrow v$ path, or ∞ if there is no such path.
- $pred(v)$ is the predecessor of v in the tentative shortest $s \rightsquigarrow v$ path, or NULL if there is no such vertex.

In fact, the predecessor pointers automatically define a tentative shortest path *tree*; they play exactly the same role as the parent pointers in our generic graph traversal algorithm. At the beginning of the algorithm, we already know that $dist(s) = 0$ and $pred(s) = \text{NULL}$. For every vertex $v \neq s$, we initially set $dist(v) = \infty$ and $pred(v) = \text{NULL}$ to indicate that we do not know of *any* path from s to v .

During the execution of the algorithm, we call an edge $u \rightarrow v$ **tense** if $dist(u) + w(u \rightarrow v) < dist(v)$. If $u \rightarrow v$ is tense, the tentative shortest path $s \rightsquigarrow v$ is clearly incorrect, because the path $s \rightsquigarrow u \rightarrow v$ is shorter. Our generic algorithm repeatedly finds a tense edge in the graph and *relaxes* it:

$\begin{array}{l} \text{RELAX}(u \rightarrow v): \\ dist(v) \leftarrow dist(u) + w(u \rightarrow v) \\ pred(v) \leftarrow u \end{array}$
--

When there are no tense edges, the algorithm halts, and we have our desired shortest path tree.

The correctness of Ford's generic relaxation algorithm follows from the following series of claims:

1. For every vertex v , the distance $dist(v)$ is either ∞ or the length of some walk from s to v . This claim can be proved by induction on the number of relaxations.
2. If the graph has no negative cycles, then $dist(v)$ is either ∞ or the length of some *simple path* from s to v . Specifically, if $dist(v)$ is the length of a walk from s to v that contains a directed cycle, that cycle must have negative weight. This claim implies that if G has no negative cycles, the relaxation algorithm eventually halts, because there are only a finite number of simple paths in G .

²Specifically, Dantzig showed that the shortest path problem can be phrased as a linear programming problem, and then described an interpretation of his simplex method in terms of the original graph. His description is equivalent to Ford's relaxation strategy.

3. If no edge in G is tense, then for every vertex v , the distance $dist(v)$ is the length of the predecessor path $s \rightarrow \dots \rightarrow pred(pred(v)) \rightarrow pred(v) \rightarrow v$. Specifically, if v violates this condition but its predecessor $pred(v)$ does not, the edge $pred(v) \rightarrow v$ is tense.
4. If no edge in G is tense, then for every vertex v , the path of predecessor edges

$$s \rightarrow \dots \rightarrow pred(pred(v)) \rightarrow pred(v) \rightarrow v$$

is a shortest path from s to v . Specifically, if v violates this condition but its predecessor u in some shortest path does not, the edge $u \rightarrow v$ is tense. This claim also implies that if the G has a negative cycle, then some edge is *always* tense, so the generic algorithm never halts.

So far I haven't said anything about how we detect which edges can be relaxed, or in what order we relax them. To make this easier, we refine the relaxation algorithm slightly, into something closely resembling the generic graph traversal algorithm. We maintain a "bag" of vertices, initially containing just the source vertex s . Whenever we take a vertex u from the bag, we scan all of its outgoing edges, looking for something to relax. Finally, whenever we successfully relax an edge $u \rightarrow v$, we put v into the bag. Unlike our generic graph traversal algorithm, we do not mark vertices when we visit them; the same vertex could be visited many times, and the same edge could be relaxed many times.

<p><u>INITSSSP(s):</u> $dist(s) \leftarrow 0$ $pred(s) \leftarrow \text{NULL}$ for all vertices $v \neq s$ $dist(v) \leftarrow \infty$ $pred(v) \leftarrow \text{NULL}$</p>	<p><u>GENERICSSSP(s):</u> INITSSSP(s) put s in the bag while the bag is not empty take u from the bag for all edges $u \rightarrow v$ if $u \rightarrow v$ is tense RELAX($u \rightarrow v$) put v in the bag</p>
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Just as with graph traversal, different "bag" data structures for the give us different algorithms. There are three obvious choices to try: a stack, a queue, and a priority queue. Unfortunately, if we use a stack, the resulting algorithm performs $\Theta(2^V)$ relaxation steps in the worst case! (Proving this is a good homework problem.) The other two possibilities are much more efficient.

8.4 Best-First: Dijkstra's Algorithm

If we implement the bag using a priority queue, where the key of a vertex v is its tentative distance $dist(v)$, we obtain an algorithm first "published" in 1957 by a team of researchers at the Case Institute of Technology, in an annual project report for the Combat Development Department of the US Army Electronic Proving Ground. The

same algorithm was later independently rediscovered and actually publicly published by Edsger Dijkstra in 1959. A nearly identical algorithm was also described by George Dantzig in 1958.

Dijkstra’s algorithm, as it is universally known³, is particularly well-behaved if the graph has no negative-weight edges. In this case, it’s not hard to show (by induction, of course) that the vertices are scanned in increasing order of their shortest-path distance from s . It follows that each vertex is scanned at most once, and thus that each edge is relaxed at most once. Since the key of each vertex in the heap is its tentative distance from s , the algorithm performs a DECREASEKEY operation every time an edge is relaxed. Thus, the algorithm performs at most E DECREASEKEYS. Similarly, there are at most V INSERT and EXTRACTMIN operations. Thus, if we store the vertices in a Fibonacci heap, the total running time of Dijkstra’s algorithm is $O(E + V \log V)$; if we use a regular binary heap, the running time is $O(E \log V)$.

This analysis assumes that no edge has negative weight. Dijkstra’s algorithm (in the form I’m presenting here⁴) is still *correct* if there are negative edges, but the worst-case running time could be exponential. (Proving this unfortunate fact is a good homework problem.) On the other hand, in practice, Dijkstra’s algorithm is usually quite fast even for graphs with negative edges.

8.5 Breadth-First: Bellman-Ford

If we replace the heap in Dijkstra’s algorithm with a FIFO queue, we obtain an algorithm first sketched by Shimbel in 1954, described in more detail by Moore in 1957, then independently rediscovered by Woodbury and Dantzig in 1957 and again by Bellman in 1958. In full compliance with Stigler’s Law, the algorithm is almost universally known as **Bellman-Ford**, because Bellman explicitly used Ford’s formulation of relaxing edges, although a few early sources refer to “Bellman-Shimbel”. Bellman-Ford is efficient even if there are negative edges, and it can be used to quickly detect the presence of negative cycles. If there are no negative edges, however, Dijkstra’s algorithm is faster. (In fact, in practice, Dijkstra’s algorithm is often faster even for graphs with negative edges.)

The easiest way to analyze the algorithm is to break the execution into *phases*, by introducing an imaginary *token*. Before we even begin, we insert the token into the queue. The current phase ends when we take the token out of the queue; we begin the

³I will follow this common convention, despite the historical inaccuracy, partly because I don’t think anybody wants to read about the “Leyzorek-Gray-Johnson-Ladew-Meaker-Petry-Seitz algorithm”, and partly because papers that aren’t actually *publically* published don’t count.

⁴Most algorithms textbooks, Wikipedia, and even Dijkstra’s original paper present a version of Dijkstra’s algorithm that gives incorrect results for graphs with negative edges, because it *never* visits the same vertex more than once. This is also the version of Dijkstra’s algorithm that I described as “best-first search” in Chapter 5. Here I’ve taken the liberty of correcting Dijkstra’s mistake. Even Dijkstra would agree that a *correct* algorithm that is sometimes slow (and in practice, *rarely* slow) is better than a fast algorithm that doesn’t always work.

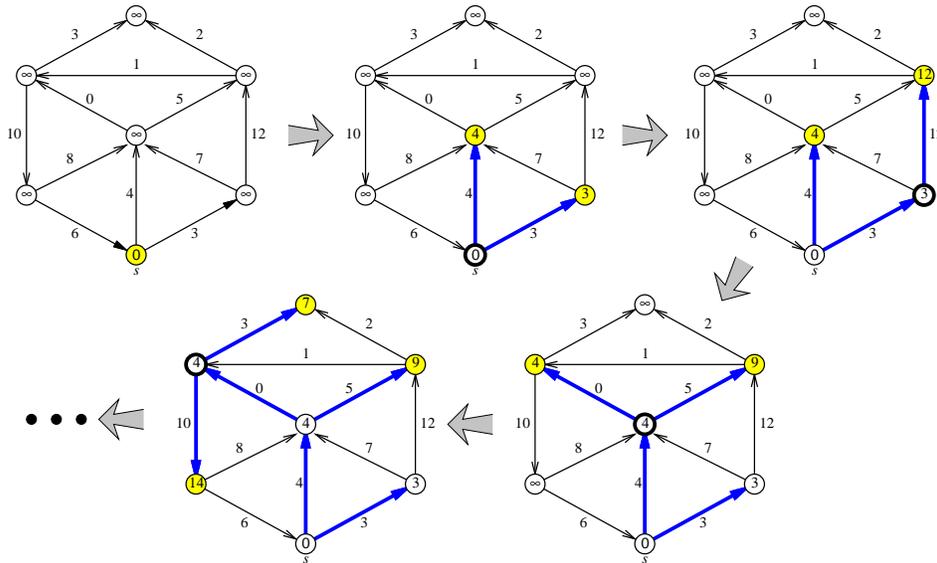


Figure 8.4. Four phases of Dijkstra's algorithm run on a graph with no negative edges. At each phase, the shaded vertices are in the heap, and the bold vertex has just been scanned. The bold edges describe the evolving shortest path tree.

next phase by reinserting the token into the queue. The 0th phase consists entirely of scanning the source vertex s . The algorithm ends when the queue contains *only* the token. A simple inductive argument (hint, hint) implies the following invariant for every integer i and vertex v :

After i phases of the algorithm, $\text{dist}(v)$ is at most the length of the shortest walk from s to v consisting of at most i edges.

Include the induction proof.



Since a shortest path can only pass through each vertex once, either the algorithm halts before the V th phase, or the graph contains a negative cycle. In each phase, we scan each vertex at most once, so we relax each edge at most once, so the running time of a single phase is $O(E)$. Thus, the overall running time of Bellman-Ford is $O(VE)$.

Once we understand how the phases of Bellman-Ford behave, we can simplify the algorithm considerably by producing the same behavior *on purpose* (following the same design pattern that we advocated for dynamic programming in Chapter 3). Instead of performing a partial breadth-first search of the graph in each phase, we can simply scan through the adjacency list directly, relaxing every tense edge we find in the graph.

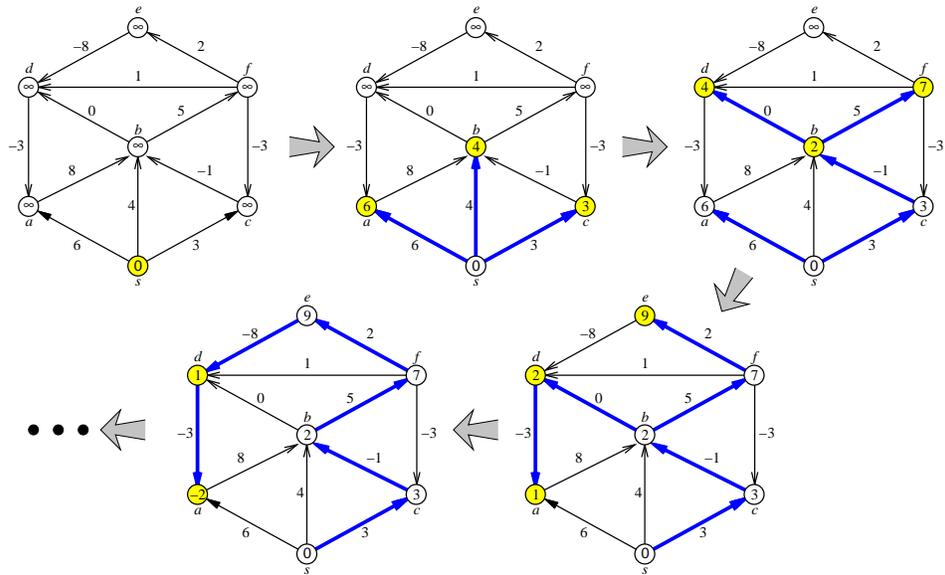


Figure 8.5. Four phases of Bellman-Ford run on a directed graph with negative edges. Nodes are taken from the queue in the order $s \diamond a b c \diamond d f b \diamond a e d \diamond d a \diamond$, where \diamond is the end-of-phase token. Shaded vertices are in the queue at the end of each phase. The bold edges describe the evolving shortest path tree.



SHIMBEL-MOORE-WOODBURY-DANTZIG-BELLMAN-FORD-BROSH:
Relax ALL the tense edges and recurse.

```

BELLMANFORDSSSP(s)
  INITSSSP(s)
  repeat V times:
    for every edge u→v
      if u→v is tense
        RELAX(u→v)
  for every edge u→v
    if u→v is tense
      return "Negative cycle!"
    
```

This is how most textbooks present the Bellman-Ford algorithm.⁵ The $O(VE)$ running

⁵In fact, this is essentially the formulation proposed by both Shimbel and Bellman. Bob Tarjan

time of this formulation of the algorithm should be obvious, but it may be less clear that the algorithm is still correct. In fact, correctness follows from exactly the same invariant as before:

After i phases of the algorithm, $dist(v)$ is at most the length of the shortest walk from s to v consisting of at most i edges.

As before, it is straightforward to prove by induction (hint, hint) that this invariant holds for every integer i and vertex v .

8.6 Dynamic Programming: Bellman-Ford Again

Like everything else with Richard Bellman's name on it, Bellman-Ford can also be recast as a dynamic programming algorithm. Let $dist_i(v)$ denote the length of the shortest path $s \rightsquigarrow v$ consisting of at most i edges. It's not hard to see that this function obeys the following Bellman's equation recurrence:

$$dist_i(v) = \begin{cases} 0 & \text{if } i = 0 \text{ and } v = s \\ \infty & \text{if } i = 0 \text{ and } v \neq s \\ \min \left\{ \begin{array}{l} dist_{i-1}(v), \\ \min_{u \rightarrow v \in E} (dist_{i-1}(u) + w(u \rightarrow v)) \end{array} \right\} & \text{otherwise} \end{cases}$$

For the moment, let's assume the graph has no negative cycles; our goal is to compute $dist_{V-1}(t)$. A straightforward dynamic-programming evaluation of this recurrence looks like this:

```
BELLMANFORDDP(s)
  dist[0, s] ← 0
  for every vertex v ≠ s
    dist[0, v] ← ∞
  for i ← 1 to V - 1
    for every vertex v
      dist[i, v] ← dist[i - 1, v]
      for every edge u → v
        if dist[i, v] > dist[i - 1, u] + w(u → v)
          dist[i, v] ← dist[i - 1, u] + w(u → v)
```

Now let us make two minor changes to this algorithm. First, we remove one level of indentation from the last three lines. This may change the order in which we examine edges, but the modified algorithm still computes $dist_i(v)$ for all i and v . Second, we

recognized in the early 1980s that Bellman-Ford is equivalent to Dijkstra's algorithm with a queue instead of a heap.

change the indices in the last two lines from $i - 1$ to i . This change may cause the distances $dist[i, v]$ to approach the true shortest-path distances more quickly than before, but the algorithm is still correct.

```

BELLMANFORDDP2(s)
  dist[0, s] ← 0
  for every vertex v ≠ s
    dist[0, v] ← ∞
  for i ← 1 to V - 1
    for every vertex v
      dist[i, v] ← dist[i - 1, v]
    for every edge u → v
      if dist[i, v] > dist[i, u] + w(u → v)
        dist[i, v] ← dist[i, u] + w(u → v)

```

Now notice that the iteration index i is completely redundant! We really only need to keep a one-dimensional array of distances, which means we don't need to scan the vertices in each iteration of the main loop.

```

BELLMANFORDDP3(s)
  dist[s] ← 0
  for every vertex v ≠ s
    dist[v] ← ∞
  for i ← 1 to V - 1
    for every edge u → v
      if dist[v] > dist[u] + w(u → v)
        dist[v] ← dist[u] + w(u → v)

```

The resulting algorithm is almost identical to our earlier algorithm `BELLMANFORDSSSP!` The first three lines initialize the shortest path distances, and the last two lines check whether an edge is tense, and if so, relaxes it. The only feature missing from the new algorithm is explicit maintenance of predecessors, but that's easy to add.

Exercises



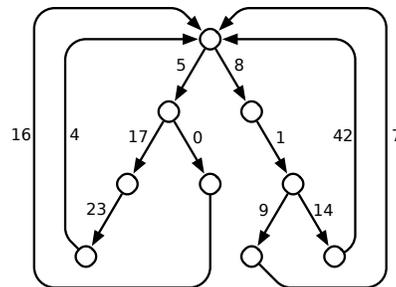
Need more!

1. Prove that the following invariant holds for every integer i and every vertex v : After i phases of Bellman-Ford (in either formulation), $dist(v)$ is at most the length of the shortest path $s \rightsquigarrow v$ consisting of at most i edges.
2. Let G be a directed graph with edge weights (which may be positive, negative, or zero), and let s be an arbitrary vertex of G .

- (a) Suppose every vertex v stores a number $dist(v)$. Describe and analyze an algorithm to determine whether $dist(v)$ is the shortest-path distance from s to v , for every vertex v .
- (b) Suppose instead that every vertex $v \neq s$ stores a pointer $pred(v)$ to another vertex in G . Describe and analyze an algorithm to determine whether these predecessor pointers define a single-source shortest path tree rooted at s .

Do **not** assume that G contains no negative cycles.

3. A *looped tree* is a weighted, directed graph built from a binary tree by adding an edge from every leaf back to the root. Every edge has a non-negative weight.



A looped tree.

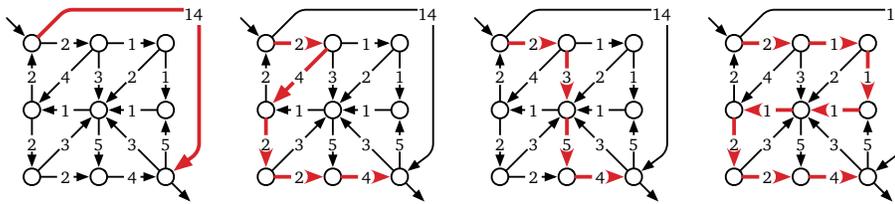
- (a) How much time would Dijkstra's algorithm require to compute the shortest path between two vertices u and v in a looped tree with n nodes?
- (b) Describe and analyze a faster algorithm.
4. Suppose we are given a directed graph G with weighted edges and two vertices s and t .
- (a) Describe and analyze an algorithm to find the shortest path from s to t when exactly one edge in G has negative weight.
- (b) Describe and analyze an algorithm to find the shortest path from s to t when exactly k edges in G have negative weight. How does the running time of your algorithm depend on k ?
5. Suppose we are given an undirected graph G in which every *vertex* has a positive weight.
- (a) Describe and analyze an algorithm to find a *spanning tree* of G with minimum total weight. (The total weight of a spanning tree is the sum of the weights of its vertices.)
- (b) Describe and analyze an algorithm to find a *path* in G from one given vertex s to another given vertex t with minimum total weight. (The total weight of a path is the sum of the weights of its vertices.)

[Hint: One of these problems is trivial.]

6. For any edge e in any graph G , let $G \setminus e$ denote the graph obtained by deleting e from G .
 - (a) Suppose we are given a directed graph G in which the shortest path σ from vertex s to vertex t passes through every vertex of G . Describe an algorithm to compute the shortest-path distance from s to t in $G \setminus e$, for every edge e of G , in $O(E \log V)$ time. Your algorithm should output a set of E shortest-path distances, one for each edge of the input graph. You may assume that all edge weights are non-negative. [Hint: If we delete an edge of the original shortest path, how do the old and new shortest paths overlap?]
 - ♥(b) Let s and t be arbitrary vertices in an arbitrary undirected graph G . Describe an algorithm to compute the shortest-path distance from s to t in $G \setminus e$, for every edge e of G , in $O(E \log V)$ time. Again, you may assume that all edge weights are non-negative.
7. Let $G = (V, E)$ be a connected directed graph with non-negative edge weights, let s and t be vertices of G , and let H be a subgraph of G obtained by deleting some edges. Suppose we want to reinsert exactly one edge from G back into H , so that the shortest path from s to t in the resulting graph is as short as possible. Describe and analyze an algorithm that chooses the best edge to reinsert, in $O(E \log V)$ time.
8. When there is more than one shortest path from one node s to another node t , it is often convenient to choose a shortest path with the fewest edges; call this the **best path** from s to t . Suppose we are given a directed graph G with positive edge weights and a source vertex s in G . Describe and analyze an algorithm to compute best paths in G from s to every other vertex.
- ♥9. (a) Prove that Ford's generic shortest-path algorithm (while the graph contains a tense edge, relax it) can take exponential time in the worst case when implemented with a stack instead of a priority queue (like Dijkstra) or a queue (like Bellman-Ford). Specifically, for every positive integer n , construct a weighted directed n -vertex graph G_n , such that the stack-based shortest-path algorithm call RELAX $\Omega(2^n)$ times when G_n is the input graph. [Hint: Towers of Hanoi.]
 - (b) Prove that Dijkstra's shortest-path algorithm can require exponential time in the worst case when edges are allowed to have negative weight. Specifically, for every positive integer n , construct a weighted directed n -vertex graph G_n , such that Dijkstra's algorithm calls RELAX $\Omega(2^n)$ times when G_n is the input graph. [Hint: This is relatively easy if you've already solved part (a).]
10. (a) Describe and analyze a modification of the Bellman-Ford shortest-path algorithm that actually returns a negative cycle if any such cycle is reachable from s , or a

shortest-path tree if there is no such cycle. The modified algorithm should still run in $O(VE)$ time.

- (b) Describe and analyze a modification of Bellman-Ford that computes the correct shortest path distances from s to every other vertex of the input graph, even if the graph contains negative cycles. Specifically, if any walk from s to v contains a negative cycle, your algorithm should end with $dist(v) = -\infty$; otherwise, $dist(v)$ should contain the length of the shortest path from s to v . The modified algorithm should still run in $O(VE)$ time.
- ♥(c) Repeat parts (a) and (b), but for Ford's generic shortest-path algorithm. You may assume that the unmodified algorithm halts in $O(2^V)$ steps if there is no negative cycle; your modified algorithms should also run in $O(2^V)$ time.
- ♥11. Although we typically speak of "the" shortest path between two nodes, a single graph could contain several minimum-length paths with the same endpoints.



Four (of many) equal-length shortest paths.

Describe and analyze an algorithm to determine the *number* of shortest paths from a source vertex s to a target vertex t in an arbitrary directed graph G with weighted edges. You may assume that all edge weights are positive and that all necessary arithmetic operations can be performed in $O(1)$ time.

[Hint: Compute shortest path distances from s to every other vertex. Throw away all edges that cannot be part of a shortest path from s to another vertex. What's left?]

12. You just discovered your best friend from elementary school on Twitbook. You both want to meet as soon as possible, but you live in two different cities that are far apart. To minimize travel time, you agree to meet at an intermediate city, and then you simultaneously hop in your cars and start driving toward each other. But where *exactly* should you meet?

You are given a weighted graph $G = (V, E)$, where the vertices V represent cities and the edges E represent roads that directly connect cities. Each edge e has a weight $w(e)$ equal to the time required to travel between the two cities. You are also given a vertex p , representing your starting location, and a vertex q , representing your friend's starting location.

Describe and analyze an algorithm to find the target vertex t that allows you and your friend to meet as quickly as possible.

13. After moving to a new city, you decide to choose a walking route from your home to your new office. To get a good daily workout, your route must consist of an uphill path (for exercise) followed by a downhill path (to cool down), or just an uphill path, or just a downhill path. (You'll walk the same path home, so you'll get exercise one way or the other.) But you also want the *shortest* path that satisfies these conditions, so that you actually get to work on time.

Your input consists of an undirected graph G , whose vertices represent intersections and whose edges represent road segments, along with a start vertex s and a target vertex t . Every vertex v has a value $h(v)$, which is the height of that intersection above sea level, and each edge uv has a value $\ell(uv)$, which is the length of that road segment.

- (a) Describe and analyze an algorithm to find the shortest uphill–downhill walk from s to t . Assume all vertex heights are distinct.
 - (b) Now suppose we allow some or all vertex heights to be equal. Describe and analyze an algorithm to find the shortest “uphill”–“downhill” walk from s to t ; you may use flat edges in both the “uphill” and “downhill” portions of your walk.
 - (c) Finally, suppose you discover that there is no path from s to t with the structure you want. Describe an algorithm to find a path from s to t that alternates between “uphill” and “downhill” subpaths as few times as possible, and has minimum length among all such paths.
14. After a grueling algorithms midterm, you decide to take the bus home. Since you planned ahead, you have a schedule that lists the times and locations of every stop of every bus in Champaign-Urbana. Unfortunately, there isn't a single bus that visits both your exam building and your home; you must transfer between bus lines at least once.

Describe and analyze an algorithm to determine the sequence of bus rides that will get you home as early as possible, assuming there are b different bus lines, and each bus stops n times per day. Your goal is to minimize your *arrival time*, not the time you actually spend traveling. Assume that the buses run exactly on schedule, that you have an accurate watch, and that you are too tired to walk between bus stops.

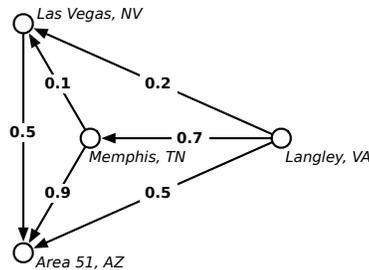
15. After graduating you accept a job with Aerophobes-Я-U's, the leading traveling agency for people who hate to fly. Your job is to build a system to help customers plan airplane trips from one city to another. All of your customers are afraid of flying (and by extension, airports), so any trip you plan needs to be as short as possible. You know all the departure and arrival times of all the flights on the planet.

Suppose one of your customers wants to fly from city X to city Y . Describe an algorithm to find a sequence of flights that minimizes the *total time in transit*—the length of time from the initial departure to the final arrival, including time at

intermediate airports waiting for connecting flights. [Hint: Modify the input data and apply Dijkstra's algorithm.]

16. Mulder and Scully have computed, for every road in the United States, the exact probability that someone driving on that road *won't* be abducted by aliens. Agent Mulder needs to drive from Langley, Virginia to Area 51, Nevada. What route should he take so that he has the least chance of being abducted?

More formally, you are given a directed graph $G = (V, E)$, where every edge e has an independent safety probability $p(e)$. The *safety* of a path is the product of the safety probabilities of its edges. Design and analyze an algorithm to determine the safest path from a given start vertex s to a given target vertex t .



For example, with the probabilities shown above, if Mulder tries to drive directly from Langley to Area 51, he has a 50% chance of getting there without being abducted. If he stops in Memphis, he has a $0.7 \times 0.9 = 63\%$ chance of arriving safely. If he stops first in Memphis and then in Las Vegas, he has a $1 - 0.7 \times 0.1 \times 0.5 = 96.5\%$ chance of being abducted! (That's how they got Elvis, you know.) Although this example is a dag, your algorithm must handle *arbitrary* directed graphs.

17. On an overnight camping trip in Sunnydale National Park, you are woken from a restless sleep by a scream. As you crawl out of your tent to investigate, a terrified park ranger runs out of the woods, covered in blood and clutching a crumpled piece of paper to his chest. As he reaches your tent, he gasps, "Get out. . . while. . . you. . .", thrusts the paper into your hands, and falls to the ground. Checking his pulse, you discover that the ranger is stone dead.

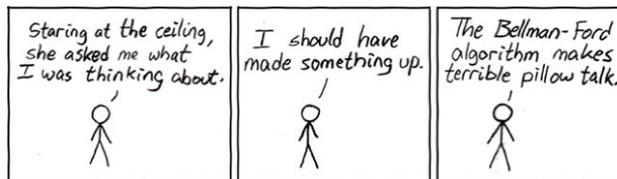
You look down at the paper and recognize a map of the park, drawn as an undirected graph, where vertices represent landmarks in the park, and edges represent trails between those landmarks. (Trails start and end at landmarks and do not cross.) You recognize one of the vertices as your current location; several vertices on the boundary of the map are labeled EXIT.

On closer examination, you notice that someone (perhaps the poor dead park ranger) has written a real number between 0 and 1 next to each vertex and each edge. A scrawled note on the back of the map indicates that a number next to an edge is the probability of encountering a vampire along the corresponding trail,

and a number next to a vertex is the probability of encountering a vampire at the corresponding landmark. (Vampires can't stand each other's company, so you'll never see more than one vampire on the same trail or at the same landmark.) The note warns you that stepping off the marked trails will result in a slow and painful death.

You glance down at the corpse at your feet. Yes, his death certainly looked painful. Wait, was that a twitch? Are his teeth getting longer? After driving a tent stake through the undead ranger's heart, you wisely decide to leave the park immediately.

Describe and analyze an efficient algorithm to find a path from your current location to an arbitrary EXIT node, such that the total *expected number* of vampires encountered along the path is as small as possible. *Be sure to account for both the vertex probabilities and the edge probabilities!*



— Randall Munroe, *xkcd* (<http://xkcd.com/69/>)
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