1. Suppose you are given a directed graph $G = (V, E)$, two vertices $s$ and $t$, a capacity function $c : E \rightarrow \mathbb{R}^+$, and a second function $f : E \rightarrow \mathbb{R}$. Describe and analyze an algorithm to determine whether $f$ is a maximum $(s,t)$-flow in $G$. [Hint: Don’t make any “obvious” assumptions!]

2. Suppose you are given a flow network $G$ with integer edge capacities and an integer maximum flow $f^*$ in $G$. Describe algorithms for the following operations:
   
   (a) $\text{INCREMENT}(e)$: Increase the capacity of edge $e$ by 1 and update the maximum flow. 
   (b) $\text{DECREMENT}(e)$: Decrease the capacity of edge $e$ by 1 and update the maximum flow. 

   Both algorithms should modify $f^*$ so that it is still a maximum flow, but more quickly than recomputing a maximum flow from scratch.

3. An $(s,t)$-series-parallel graph is a directed acyclic graph with two designated vertices $s$ (the source) and $t$ (the target or sink) and with one of the following structures:
   
   - **Base case**: A single directed edge from $s$ to $t$. 
   - **Series**: The union of an $(s,u)$-series-parallel graph and a $(u,t)$-series-parallel graph that share a common vertex $u$ but no other vertices or edges. 
   - **Parallel**: The union of two smaller $(s,t)$-series-parallel graphs with the same source $s$ and target $t$, but with no other vertices or edges in common.

   Every $(s,t)$-series-parallel graph $G$ can be represented by a decomposition tree, which is a binary tree with three types of nodes: leaves corresponding to single edges in $G$, series nodes (each labeled by some vertex), and parallel nodes (unlabeled).

   (a) Suppose you are given a directed graph $G$ with two special vertices $s$ and $t$. Describe and analyze an algorithm that either builds a decomposition tree for $G$ or correctly reports that $G$ is not $(s,t)$-series-parallel. [Hint: Build the tree from the bottom up.]

   (b) Describe and analyze an algorithm to compute a maximum $(s,t)$-flow in a given $(s,t)$-series-parallel flow network with arbitrary edge capacities. [Hint: In light of part (a), you can assume that you are actually given the decomposition tree.]