Balanced binary search trees

Goal: $O(\log n)$ time per op

C++ STL red-black tree \[\text{complicated}\]

treap Aragon Seidel '90

binary tree: every node has a key and a priority

Binary search tree \[\text{min-heap}\]

Unique for any given keys + priorities

Given an arbitrary set of keys

find priorities s.t. treap is balanced

Priorities are random uniform real numbers in $[0,1]$

\[\Rightarrow\] unique with prob. 1.
treap = recursion tree for quicksort!
Insert(x):
Assign x a random priority
BST insert(x) <= [depth]
while pri(x) < pri(parent(x))
  rotate(x)

Delete(x):
  Insert backwards in time
  pri(x) < infinity
while (x is not a leaf)
  if pri(left(x)) < pri(right(x))
    rotate(left(x))
  else
    rotate(right(x))
Time for each op is depth of some node.

Claim: For every key $x$, if all priorities are random, $E[\text{depth}(x)] = O(\log n)$

$E[\text{depth}] = E[\# \text{proper ancestors}]$

$= \sum_v \Pr[v \text{ is a proper ancestor}]$

$[i \uparrow k] = 1$ if node with $i^{th}$ smallest key is proper ancestor of the node with $k^{th}$ smallest key

$\Pr[i \uparrow k] = \Pr[\text{randomized quicksort compares } i:k]$ when $i$ is pivot.
$$X(i,k) = \sum_{i} i, i+1, \ldots, k^3 \text{ or } \sum_{k,k+1,\ldots,i^3}$$

Claim: \([i \uparrow k] \iff \text{smallest priority in } X(i,k) \text{ is node } i\]

Proof: 

$$\Pr[i \uparrow k] = \Pr[i \text{ is smallest in } X(i,k)]$$

$$= \frac{1}{|X(i,k)|} = \frac{1}{|k-i| + 1}$$

$$E[\text{depth}(i)] = \sum_k \Pr[k \uparrow i]$$

$$= \sum_{k \leq i} \frac{1}{k-i+1} + \sum_{k > i} \frac{1}{k-i+1}$$

$$\leq 2H_n$$

$$\leq 2 \ln(n+1)$$
For all $x$, $\mathbb{E}([\text{depth}(x)]) = O(\log n)$

$\mathbb{E}([\text{max depth}]) = O(\log n)$

$\Pr[\text{max depth} > 4 \ln n]$:

- Level is bad if biggest subproblem is split badly

$\Pr[\text{level is bad}] = \frac{1}{2}$

Max # good levels $\leq \log_{4/3} n$
\[ \Pr \left[ \# \text{bad levels} \geq 3 \log n \right] \]

\[ = \Pr \left[ \text{flip coin } 4 \log n \text{ times} \Rightarrow 3 \log n \text{ tails} \right] \]

Tail inequality: \[ \Pr \left[ X > \alpha \mathbb{E}[X] \right] \leq ? \]