A process cannot be understood by stopping it. Understanding must move with the flow of the process, must join it and flow with it.

— The First Law of Mentat, in Frank Herbert's *Dune* (1965)

Contrary to expectation, flow usually happens not during relaxing moments of leisure and entertainment, but rather when we are actively involved in a difficult enterprise, in a task that stretches our mental and physical abilities. . . . Flow is hard to achieve without effort. Flow is not "wasting time."

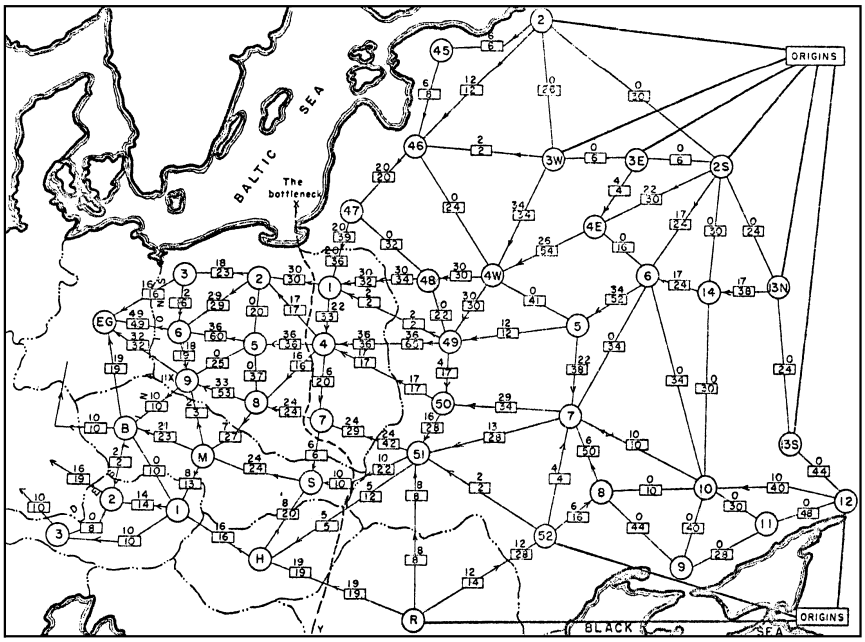
— Mihaly Csíkszentmihályi, *Flow: The Psychology of Optimal Experience* (1990)

There's a difference between knowing the path and walking the path.

— Morpheus [Laurence Fishburne], *The Matrix* (1999)

23 Maximum Flows and Minimum Cuts

In the mid-1950s, Air Force researcher Theodore E. Harris and retired army general Frank S. Ross published a classified report studying the rail network that linked the Soviet Union to its satellite countries in Eastern Europe. The network was modeled as a graph with 44 vertices, representing geographic regions, and 105 edges, representing links between those regions in the rail network. Each edge was given a weight, representing the rate at which material could be shipped from one region to the next. Essentially by trial and error, they determined both the maximum amount of stuff that could be moved from Russia into Europe, as well as the cheapest way to disrupt the network by removing links (or in less abstract terms, blowing up train tracks), which they called 'the bottleneck'. Their results, including the drawing of the network below, were only declassified in 1999.[1]



Harris and Ross's map of the Warsaw Pact rail network

[1]Both the map and the story were taken from Alexander Schrijver's fascinating survey 'On the history of combinatorial optimization (till 1960)'.

This one of the first recorded applications of the *maximum flow* and *minimum cut* problems. For both problems, the input is a directed graph $G = (V, E)$, along with special vertices s and t called the *source* and *target*. As in the previous lectures, I will use $u \to v$ to denote the directed edge from vertex u to vertex v. Intuitively, the maximum flow problem asks for the largest amount of material that can be transported from s to t; the minimum cut problem asks for the minimum damage needed to separate s from t.

23.1 Flows

An ***(s, t)-flow*** (or just a ***flow*** if the source and target are clear from context) is a function $f : E \to \mathbb{R}_{\geq 0}$ that satisfies the following ***conservation constraint*** at every vertex v except possibly s and t:

$$\sum_u f(u \to v) = \sum_w f(v \to w).$$

In English, the total flow into v is equal to the total flow out of v. To keep the notation simple, we define $f(u \to v) = 0$ if there is no edge $u \to v$ in the graph. The ***value*** of the flow f, denoted $|f|$, is the total net flow out of the source vertex s:

$$|f| := \sum_w f(s \to w) - \sum_u f(u \to s).$$

It's not hard to prove that $|f|$ is also equal to the total net flow *into* the target vertex t, as follows. To simplify notation, let $\partial f(v)$ denote the total net flow out of any vertex v:

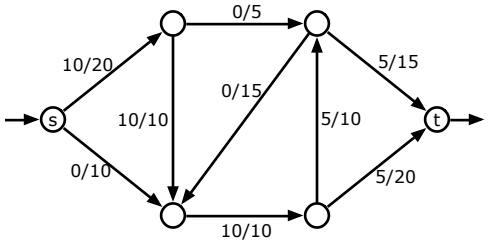
$$\partial f(v) := \sum_u f(u \to v) - \sum_w f(v \to w).$$

The conservation constraint implies that $\partial f(v) = 0$ or every vertex v except s and t, so

$$\sum_v \partial f(v) = \partial f(s) + \partial f(t).$$

On the other hand, any flow that leaves one vertex must enter another vertex, so we must have $\sum_v \partial f(v) = 0$. It follows immediately that $|f| = \partial f(s) = -\partial f(t)$.

Now suppose we have another function $c : E \to \mathbb{R}_{\geq 0}$ that assigns a non-negative ***capacity*** $c(e)$ to each edge e. We say that a flow f is ***feasible*** (with respect to c) if $f(e) \leq c(e)$ for every edge e. Most of the time we will consider only flows that are feasible with respect to some fixed capacity function c. We say that a flow f ***saturates*** edge e if $f(e) = c(e)$, and ***avoids*** edge e if $f(e) = 0$. The ***maximum flow problem*** is to compute a feasible (s, t)-flow in a given directed graph, with a given capacity function, whose value is as large as possible.



An (s, t)-flow with value 10. Each edge is labeled with its flow/capacity.

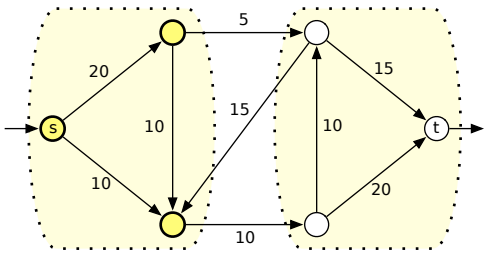
23.2 Cuts

An ***(s, t)-cut*** (or just ***cut*** if the source and target are clear from context) is a partition of the vertices into disjoint subsets S and T—meaning $S \cup T = V$ and $S \cap T = \emptyset$—where $s \in S$ and $t \in T$.

If we have a capacity function $c : E \to \mathbb{R}_{\geq 0}$, the ***capacity*** of a cut is the sum of the capacities of the edges that start in S and end in T:

$$\|S, T\| := \sum_{v \in S} \sum_{w \in T} c(v \to w).$$

(Again, if $v \to w$ is not an edge in the graph, we assume $c(v \to w) = 0$.) Notice that the definition is asymmetric; edges that start in T and end in S are unimportant. The ***minimum cut problem*** is to compute an (s, t)-cut whose capacity is as large as possible.



An (s, t)-cut with capacity 15. Each edge is labeled with its capacity.

Intuitively, the minimum cut is the cheapest way to disrupt all flow from s to t. Indeed, it is not hard to show the following relationship between flows and cuts:

Lemma 1. *Let f be **any** feasible (s, t)-flow, and let (S, T) be **any** (s, t)-cut. The value of f is at most the capacity of (S, T). Moreover, $|f| = \|S, T\|$ if and only if f saturates every edge from S to T and avoids every edge from T to S.*

Proof: Choose your favorite flow f and your favorite cut (S, T), and then follow the bouncing inequalities:

$$\begin{aligned}
|f| &= \sum_w f(s \to w) - \sum_u f(u \to s) && \text{by definition} \\
&= \sum_{v \in S} \left(\sum_w f(v \to w) - \sum_u f(u \to v) \right) && \text{by the conservation constraint} \\
&= \sum_{v \in S} \left(\sum_{w \in T} f(v \to w) - \sum_{u \in T} f(u \to v) \right) && \text{removing duplicate edges} \\
&\leq \sum_{v \in S} \sum_{w \in T} f(v \to w) && \text{because } f(u \to v) \geq 0 \\
&\leq \sum_{v \in S} \sum_{w \in T} c(v \to w) && \text{because } f(v \to w) \leq c(v \to w) \\
&= \|S, T\| && \text{by definition}
\end{aligned}$$

The two inequalities in this derivation are actually equalities if and only if $f(u \to v) = 0$ and $f(v \to w) = c(v \to w)$ for all $v \in S$ and $u, w \in T$. □

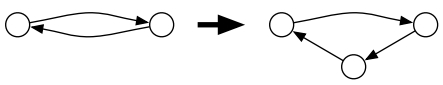
Lemma 1 implies that if $|f| = \|S, T\|$, then f must be a maximum flow, and (S, T) must be a minimum cut.

23.3 The Maxflow Mincut Theorem

Surprisingly, for any weighted directed graph, there is always a flow f and a cut (S, T) that satisfy the equality condition. This is the famous *max-flow min-cut theorem*, first proved by Lester Ford (of shortest path fame) and Delbert Ferguson in 1954 and independently by Peter Elias, Amiel Feinstein, and and Claude Shannon (of information theory fame) in 1956.

The Maxflow Mincut Theorem. *In any flow network with source s and target t, the value of the maximum (s,t)-flow is* equal *to the capacity of the minimum (s,t)-cut.*

Ford and Fulkerson proved this theorem as follows. Fix a graph G, vertices s and t, and a capacity function $c : E \to \mathbb{R}_{\geq 0}$. The proof will be easier if we assume that the capacity function is ***reduced***: For any vertices u and v, either $c(u{\to}v) = 0$ or $c(v{\to}u) = 0$, or equivalently, if an edge appears in G, then its reversal does not. This assumption is easy to enforce. Whenever an edge $u{\to}v$ and its reversal $v{\to}u$ are both the graph, replace the edge $u{\to}v$ with a path $u{\to}x{\to}v$ of length two, where x is a new vertex and $c(u{\to}x) = c(x{\to}v) = c(u{\to}v)$. The modified graph has the same maximum flow value and minimum cut capacity as the original graph.

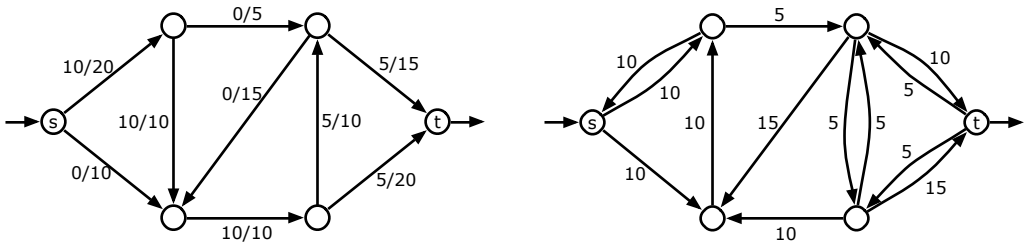


Enforcing the one-direction assumption.

Let f be a feasible flow. We define a new capacity function $c_f : V \times V \to \mathbb{R}$, called the ***residual capacity***, as follows:

$$c_f(u{\to}v) = \begin{cases} c(u{\to}v) - f(u{\to}v) & \text{if } u{\to}v \in E \\ f(v{\to}u) & \text{if } v{\to}u \in E \\ 0 & \text{otherwise} \end{cases}.$$

Since $f \geq 0$ and $f \leq c$, the residual capacities are always non-negative. It is possible to have $c_f(u{\to}v) > 0$ even if $u{\to}v$ is not an edge in the original graph G. Thus, we define the ***residual graph*** $G_f = (V, E_f)$, where E_f is the set of edges whose residual capacity is positive. Notice that the residual capacities are *not* necessarily reduced; it is quite possible to have both $c_f(u{\to}v) > 0$ and $c_f(v{\to}u) > 0$.

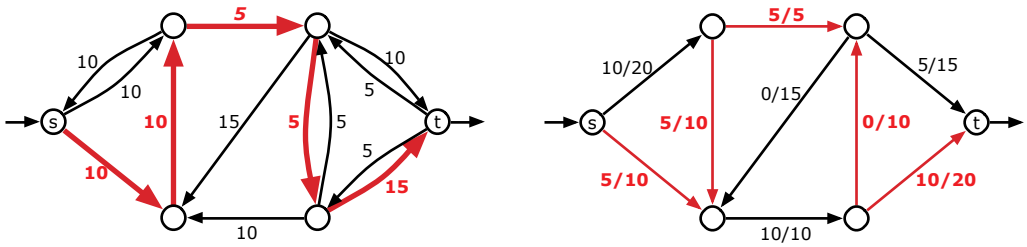


A flow f in a weighted graph G and the corresponding residual graph G_f.

Suppose there is no path from the source s to the target t in the residual graph G_f. Let S be the set of vertices that are reachable from s in G_f, and let $T = V \setminus S$. The partition (S, T) is clearly an (s,t)-cut. For every vertex $u \in S$ and $v \in T$, we have

$$c_f(u{\to}v) = (c(u{\to}v) - f(u{\to}v)) + f(v{\to}u) = 0,$$

which implies that $c(u\to v) - f(u\to v) = 0$ and $f(v\to u) = 0$. In other words, our flow f saturates every edge from S to T and avoids every edge from T to S. Lemma 1 now implies that $|f| = \|S, T\|$, which means f is a maximum flow and (S, T) is a minimum cut.



An augmenting path in G_f with value $F = 5$ and the augmented flow f'.

On the other hand, suppose there is a path $s = v_0\to v_1\to\cdots\to v_r = t$ in G_f. We refer to $v_0\to v_1\to\cdots\to v_r$ as an ***augmenting path***. Let $F = \min_i c_f(v_i\to v_{i+1})$ denote the maximum amount of flow that we can push through the augmenting path in G_f. We define a new flow function $f' : E \to \mathbb{R}$ as follows:

$$f'(u\to v) = \begin{cases} f(u\to v) + F & \text{if } u\to v \text{ is in the augmenting path} \\ f(u\to v) - F & \text{if } v\to u \text{ is in the augmenting path} \\ f(u\to v) & \text{otherwise} \end{cases}$$

To prove that the flow f' is feasible with respect to the original capacities c, we need to verify that $f' \geq 0$ and $f' \leq c$. Consider an edge $u\to v$ in G. If $u\to v$ is in the augmenting path, then $f'(u\to v) > f(u\to v) \geq 0$ and

$$\begin{aligned} f'(u\to v) &= f(u\to v) + F && \text{by definition of } f' \\ &\leq f(u\to v) + c_f(u\to v) && \text{by definition of } F \\ &= f(u\to v) + c(u\to v) - f(u\to v) && \text{by definition of } c_f \\ &= c(u\to v) && \text{Duh.} \end{aligned}$$

On the other hand, if the reversal $v\to u$ is in the augmenting path, then $f'(u\to v) < f(u\to v) \leq c(u\to v)$, which implies that

$$\begin{aligned} f'(u\to v) &= f(u\to v) - F && \text{by definition of } f' \\ &\geq f(u\to v) - c_f(v\to u) && \text{by definition of } F \\ &= f(u\to v) - f(u\to v) && \text{by definition of } c_f \\ &= 0 && \text{Duh.} \end{aligned}$$

Finally, we observe that (without loss of generality) only the first edge in the augmenting path leaves s, so $|f'| = |f| + F > 0$. In other words, f is *not* a maximum flow.

This completes the proof!

23.4 Ford and Fulkerson's augmenting-path algorithm

Ford and Fulkerson's proof of the Maxflow-Mincut Theorem translates immediately to an algorithm to compute maximum flows: Starting with the zero flow, repeatedly augment the flow along **any** path from s to t in the residual graph, until there is no such path.

This algorithm has an important but straightforward corollary:

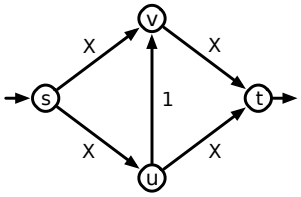
Integrality Theorem. *If all capacities in a flow network are integers, then there is a maximum flow such that the flow through every edge is an integer.*

Proof: We argue by induction that after each iteration of the augmenting path algorithm, all flow values and residual capacities are integers. Before the first iteration, residual capacities are the original capacities, which are integral by definition. In each later iteration, the induction hypothesis implies that the capacity of the augmenting path is an integer, so augmenting changes the flow on each edge, and therefore the residual capacity of each edge, by an integer.

In particular, the algorithm increases the overall value of the flow by a positive integer, which implies that the augmenting path algorithm halts and returns a maximum flow. □

If every edge capacity is an integer, the algorithm halts after $|f^*|$ iterations, where f^* is the actual maximum flow. In each iteration, we can build the residual graph G_f and perform a whatever-first-search to find an augmenting path in $O(E)$ time. Thus, for networks with integer capacities, the Ford-Fulkerson algorithm runs in $\boldsymbol{O(E|f^*|)}$ time in the worst case.

The following example shows that this running time analysis is essentially tight. Consider the 4-node network illustrated below, where X is some large integer. The maximum flow in this network is clearly $2X$. However, Ford-Fulkerson might alternate between pushing 1 unit of flow along the augmenting path $s \to u \to v \to t$ and then pushing 1 unit of flow along the augmenting path $s \to v \to u \to t$, leading to a running time of $\Theta(X) = \Omega(|f^*|)$.



A bad example for the Ford-Fulkerson algorithm.

Ford and Fulkerson's algorithm works quite well in many practical situations, or in settings where the maximum flow value $|f^*|$ is small, but without further constraints on the augmenting paths, this is *not* an efficient algorithm in general. The example network above can be described using only $O(\log X)$ bits; thus, the running time of Ford-Fulkerson is actually *exponential* in the input size.

23.5 Irrational Capacities

If we multiply all the capacities by the same (positive) constant, the maximum flow increases everywhere by the same constant factor. It follows that if all the edge capacities are *rational*, then the Ford-Fulkerson algorithm eventually halts, although still in exponential time.

However, if we allow *irrational* capacities, the algorithm can actually loop forever, always finding smaller and smaller augmenting paths! Worse yet, this infinite sequence of augmentations may not even converge to the maximum flow, or even to a significant fraction of the maximum flow! Perhaps the simplest example of this effect was discovered by Uri Zwick.

Consider the six-node network shown on the next page. Six of the nine edges have some large integer capacity X, two have capacity 1, and one has capacity $\phi = (\sqrt{5}-1)/2 \approx 0.618034$, chosen so that $1 - \phi = \phi^2$. To prove that the Ford-Fulkerson algorithm can get stuck, we can watch the residual capacities of the three horizontal edges as the algorithm progresses. (The residual capacities of the other six edges will always be at least $X - 3$.)

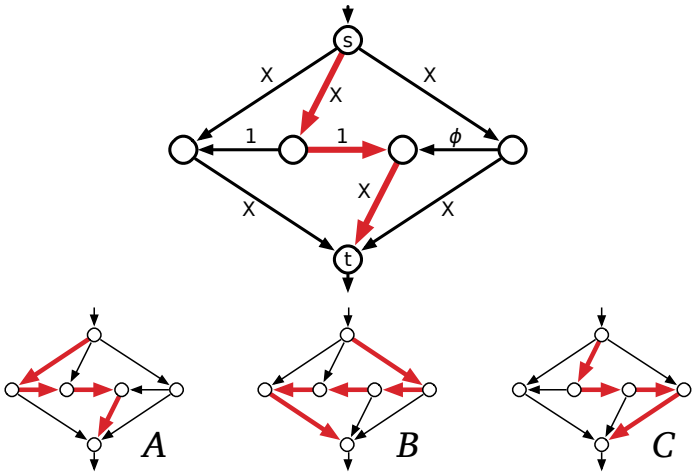
Suppose the Ford-Fulkerson algorithm starts by choosing the central augmenting path, shown in the large figure on the next page. The three horizontal edges, in order from left to right, now have residual capacities 1, 0, and ϕ. Suppose inductively that the horizontal residual capacities are ϕ^{k-1}, 0, ϕ^k for some non-negative integer k.

1. Augment along B, adding ϕ^k to the flow; the residual capacities are now $\phi^{k+1}, \phi^k, 0$.
2. Augment along C, adding ϕ^k to the flow; the residual capacities are now $\phi^{k+1}, 0, \phi^k$.
3. Augment along B, adding ϕ^{k+1} to the flow; the residual capacities are now $0, \phi^{k+1}, \phi^{k+2}$.
4. Augment along A, adding ϕ^{k+1} to the flow; the residual capacities are now $\phi^{k+1}, 0, \phi^{k+2}$.

It follows by induction that after $4n+1$ augmentation steps, the horizontal edges have residual capacities $\phi^{2n-2}, 0, \phi^{2n-1}$. As the number of augmentations grows to infinity, the value of the flow converges to

$$1 + 2\sum_{i=1}^{\infty} \phi^i = 1 + \frac{2}{1-\phi} = 4 + \sqrt{5} < 7,$$

even though the maximum flow value is clearly $2X + 1 \gg 7$.



Uri Zwick's non-terminating flow example, and three augmenting paths.

Practically-minded readers might wonder why anyone should care about irrational capacities; after all, computers can't represent anything but (small) integers or (dyadic) rationals exactly. Good question! The mathematician's answer is that the restriction to integer capacities is literally *artificial*; it's an *artifact* of physical computational hardware (or perhaps the otherwise-irrelevant laws of physics), not an inherent feature of the abstract computational problem. But a more practical reason is that the behavior of the algorithm with irrational inputs tells us something about its worst-case behavior *in practice* given floating-point capacities—terrible! Even with very reasonable capacities, a careless implementation of Ford-Fulkerson could enter an infinite loop, simply because of round-off error, without ever coming close to the correct answer.

23.6 Combining and Decomposing Flows

Flows are normally defined as functions on the edges of a graph satisfying certain constraints at the vertices. However, flows have a second representation that is more natural and useful in certain contexts.

Consider an arbitrary graph G with source vertex s and target vertex t. Fix any two (s,t)-flows f and g and any two real numbers α and β, and consider the function $h: E \to \mathbb{R}$ defined by setting

$$h(u{\to}v) := \alpha \cdot f(u{\to}v) + \beta \cdot g(u{\to}v)$$

for every edge $u{\to}v$; we can write this definition more simply as $h = \alpha f + \beta g$. Straightforward definition-chasing implies that h is also an (s,t)-flow with value $|h| = \alpha|f| + \beta|g|$. More generally, any weighted sum of (s,t)-flows is also an (s,t)-flow.

It turns out that any (s,t)-flow can be written as a weighted sum of flows with a very special structure. For any directed path P from s to t, we define a corresponding ***path flow*** as follows;

$$P(u{\to}v) = \begin{cases} 1 & \text{if } u{\to}v \in P, \\ -1 & \text{if } v{\to}u \in P, \\ 0 & \text{otherwise.} \end{cases}$$

Straightforward definition-chasing implies that the function $P: E \to \mathbb{R}$ is indeed an (s,t)-flow with value 1. I am deliberately overloading the variable P to mean both the path (a sequence of vertices and directed edges) and the unit flow along that path. Similarly, for any directed cycle C, we define a corresponding ***cycle flow***

$$C(u{\to}v) = \begin{cases} 1 & \text{if } u{\to}v \in C, \\ -1 & \text{if } v{\to}u \in C, \\ 0 & \text{otherwise;} \end{cases}$$

again, it is easy to verify that $C: E \to \mathbb{R}$ is an (s,t)-flow with value zero. Our earlier argument implies that any weighted sum of these path and cycle flows gives us another an (s,t)-flow; this weighted sum is called a ***flow decomposition*** of the resulting flow. Moreover, *every* flow has such a decomposition.

Flow Decomposition Theorem. *Every feasible (s,t)-flow f can be written as a weighted sum of directed (s,t)-paths and directed cycles. Moreover, a directed edge $u{\to}v$ appears in at least one of these paths or cycles if and only if $f(u{\to}v) > 0$, and the total number of paths and cycles is at most the number of edges in the network.*

Proof: We prove the theorem by induction on the number of edges carrying non-zero flow. Fix an arbitrary (s,t)-flow f in an arbitrary flow network G. There are three cases to consider:

- If $f(u{\to}v) = 0$ for every edge $u{\to}v$, then f is a weighted sum of the *empty* set of paths and cycles.
- Suppose $|f| = 0$, meaning flow is conserved at *every* vertex, including s and t. Pick an arbitrary edge $u{\to}v$ with $f(u{\to}v) > 0$. Consider an arbitrary walk $v_0{\to}v_1{\to}v_2{\to}\cdots$ with $v_0 = u$ and $v_1 = v$, such that $f(v_{i-1}{\to}v_i) > 0$ for every index i. The conservation constraint implies that every vertex that has incoming flow also has outgoing flow, so we can make this walk arbitrarily long; in particular, the walk must eventually visit some vertex more than once. Let $j < k$ be the smallest indices such that $v_j = v_k$. Then the subwalk $v_j{\to}v_{j-1}{\to}\cdots{\to}v_k$ is actually a simple directed cycle C.

Define $f_{\min}(C) := \min_{e \in C} f(e)$, and consider the function $f' := f - f_{\min}(C) \cdot C$, or more verbosely,

$$f'(u \to v) := \begin{cases} f(u \to v) - f_{\min}(C) & \text{if } u \to v \in C, \\ f(u \to v) + f_{\min}(C) & \text{if } v \to u \in C, \\ f(u \to v) & \text{otherwise.} \end{cases}$$

Straightforward definition chasing shows that f' is indeed a feasible flow in G with value 0. There is at least one edge $e \in C$ such that $f(e) = f_{\min}(C)$ and therefore $f'(e) = 0$. Thus, fewer edges carry flow in f' than in f. The induction hypothesis implies that f' has a valid decomposition into at most $E - 1$ paths and cycles. Adding C with the appropriate weight gives us a flow decomposition for f; specifically, $f = f' + f_{\min}(C) \cdot C$.

- The final case $|f| > 0$ is similar to the previous case. Conservation implies that there is a directed walk $s \to v_1 \to v_2 \to \cdots \to v_\ell \to t$ where every edge carries positive flow. By removing loops, we can assume without loss of generality that this walk is a simple path P.

  Let $f_{\min}(P) := \min_{e \in P} f(e)$; and define a new flow $f' := f - f_{\min}(P) \cdot P$. We easily verify that f' is a feasible flow in G with value $|f| - f_{\min}(P)$. The induction hypothesis implies that f' can be decomposed into at most $E - 1$ paths and cycles. Adding P with weight gives us a flow decomposition for f.

In all cases, we obtain a valid flow decomposition. □

The proof of the Flow Decomposition Theorem implies stronger results in two interesting special cases. First, we can decompose any flow with value zero into a weighted sum of cycles; no paths are necessary. Flows with value zero are often called ***circulations***. On the other hand, we can decompose any acyclic (s, t)-flow into a weighted sum of (s, t)-paths; no cycles are necessary.

The proof also immediately translates directly into an algorithm, similar to the Ford-Fulkerson algorithm, to decompose any (s, t)-flow into paths and cycles. The algorithm repeatedly seeks either a directed (s, t)-path or a directed cycle in the remaining flow, and then subtracts as much flow as possible along that path or cycle, until the flow is empty. Each iteration can be executed in $O(V)$ time and removes at least one edge from the graph, so the entire flow-decomposition algorithm runs in $\boldsymbol{O(VE)}$ ***time***.

Flow decompositions provide a natural lower bound on the running time of any maximum-flow algorithm that builds the flow one path or cycle at a time. Every flow can be decomposed into at most E paths and cycles, each of which uses at most V edges, so the overall complexity of the flow decomposition is $O(VE)$. Moreover, it is easy to construct flows for which every flow decomposition has complexity $\Theta(VE)$. Thus, any maximum-flow algorithm that (either explicitly or implicitly) constructs a flow as a sum of paths or cycles—in particular, any implementation of Ford and Fulkerson's augmenting path algorithm—must require $\Omega(VE)$ time in the worst case.

23.7 Edmonds and Karp's Algorithms

Ford and Fulkerson's algorithm does not specify which path in the residual graph to augment, and the poor behavior of the algorithm can be blamed on poor choices for the augmenting path. In the early 1970s, Jack Edmonds and Richard Karp analyzed two natural rules for choosing augmenting paths, both of which led to more efficient algorithms.

23.7.1 Fat Pipes

Edmonds and Karp's first rule is essentially a greedy algorithm:

> Choose the augmenting path with largest bottleneck value.

It's a fairly easy to show that the maximum-bottleneck (s, t)-path in a directed graph can be computed in $O(E \log V)$ time using a variant of Jarník's minimum-spanning-tree algorithm, or of Dijkstra's shortest path algorithm. Simply grow a directed spanning tree T, rooted at s. Repeatedly find the highest-capacity edge leaving T and add it to T, until T contains a path from s to t. Alternately, one could emulate Kruskal's algorithm—insert edges one at a time in decreasing capacity order until there is a path from s to t—although this is less efficient, at least when the graph is directed.

We can now analyze the algorithm in terms of the value of the maximum flow f^*. Let f be any flow in G, and let f' be the maximum flow *in the current residual graph* G_f. (At the beginning of the algorithm, $G_f = G$ and $f' = f^*$.) We have already proved that f' can be decomposed into at most E paths and cycles. A simple averaging argument implies that at least one of the paths in this decomposition must carry at least $|f'|/E$ units of flow. It follows immediately that the *fattest* (s, t)-path in G_f carries at least $|f'|/E$ units of flow.

Thus, augmenting f along the maximum-bottleneck path in G_f multiplies the value of the remaining maximum flow in G_f by a factor of at most $1 - 1/E$. In other words, the residual maximum flow value *decays exponentially* with the number of iterations. After $E \cdot \ln|f^*|$ iterations, the maximum flow value in G_f is at most

$$|f^*| \cdot (1 - 1/E)^{E \cdot \ln|f^*|} < |f^*| \, e^{-\ln|f^*|} = 1.$$

(That's Euler's constant e, not the edge e. Sorry.) In particular, *if all the capacities are integers*, then after $E \cdot \ln|f^*|$ iterations, the maximum capacity of the residual graph is *zero* and f is a maximum flow.

We conclude that for graphs with integer capacities, the Edmonds-Karp 'fat pipe' algorithm runs in $\boldsymbol{O(E^2 \log E \log|f^*|)}$ time, which is actually a polynomial function of the input size.

23.7.2 Short Pipes

The second Edmonds-Karp rule was actually proposed by Ford and Fulkerson in their original max-flow paper; a variant of this rule was independently considered by the Russian mathematician Yefim Dinits around the same time as Edmonds and Karp.

> Choose the augmenting path with the smallest number of edges.

The shortest augmenting path can be found in $O(E)$ time by running breadth-first search in the residual graph. Surprisingly, the resulting algorithm halts after a polynomial number of iterations, independent of the actual edge capacities!

The proof of this polynomial upper bound relies on two observations about the evolution of the residual graph. Let f_i be the current flow after i augmentation steps, let G_i be the corresponding residual graph. In particular, f_0 is zero everywhere and $G_0 = G$. For each vertex v, let $level_i(v)$ denote the unweighted shortest path distance from s to v in G_i, or equivalently, the *level* of v in a breadth-first search tree of G_i rooted at s.

Our first observation is that these levels can only increase over time.

Lemma 2. $\text{level}_{i+1}(v) \geq \text{level}_i(v)$ *for all vertices* v *and integers* i.

Proof: The claim is trivial for $v = s$, since $level_i(s) = 0$ for all i. Choose an arbitrary vertex $v \neq s$, and let $s \to \cdots \to u \to v$ be a shortest path from s to v in G_{i+1}. (If there is no such path, then $level_{i+1}(v) = \infty$, and we're done.) Because this is a shortest path, we have $level_{i+1}(v) = level_{i+1}(u) + 1$, and the inductive hypothesis implies that $level_{i+1}(u) \geq level_i(u)$.

We now have two cases to consider. If $u \to v$ is an edge in G_i, then $level_i(v) \leq level_i(u) + 1$, because the levels are defined by breadth-first traversal.

On the other hand, if $u \to v$ is not an edge in G_i, then $v \to u$ must be an edge in the ith augmenting path. Thus, $v \to u$ must lie on the shortest path from s to t in G_i, which implies that $level_i(v) = level_i(u) - 1 \leq level_i(u) + 1$.

In both cases, we have $level_{i+1}(v) = level_{i+1}(u) + 1 \geq level_i(u) + 1 \geq level_i(v)$. □

Whenever we augment the flow, the bottleneck edge in the augmenting path disappears from the residual graph, and some other edge in the *reversal* of the augmenting path may (re-)appear. Our second observation is that an edge cannot appear or disappear too many times.

Lemma 3. *During the execution of the Edmonds-Karp short-pipe algorithm, any edge* $u \to v$ *disappears from the residual graph* G_f *at most* $V/2$ *times.*

Proof: Suppose $u \to v$ is in two residual graphs G_i and G_{j+1}, but not in any of the intermediate residual graphs $G_{i+1}, \ldots, G_j$, for some $i < j$. Then $u \to v$ must be in the ith augmenting path, so $level_i(v) = level_i(u) + 1$, and $v \to u$ must be on the jth augmenting path, so $level_j(v) = level_j(u) - 1$. By the previous lemma, we have

$$level_j(u) = level_j(v) + 1 \geq level_i(v) + 1 = level_i(u) + 2.$$

In other words, the distance from s to u increased by at least 2 between the disappearance and reappearance of $u \to v$. Since every level is either less than V or infinite, the number of disappearances is at most $V/2$. □

Now we can derive an upper bound on the number of iterations. Since each edge can disappear at most $V/2$ times, there are at most $EV/2$ edge disappearances overall. But at least one edge disappears on each iteration, so the algorithm must halt after at most $EV/2$ iterations. Finally, since each iteration requires $O(E)$ time, this algorithm runs in $\boldsymbol{O(VE^2)}$ ***time*** overall.

23.8 Further Progress

This is nowhere near the end of the story for maximum-flow algorithms. Decades of further research have led to a number of even faster algorithms, some of which are summarized in the table below.[2] All of the algorithms listed below compute a maximum flow in several iterations. Each algorithm has two variants: a simpler version that performs each iteration by brute force, and a faster variant that uses sophisticated data structures to maintain a spanning tree of the flow network, so that each iteration can be performed (and the spanning tree updated) in logarithmic time. There is no reason to believe that the best algorithms known so far are optimal; indeed, maximum flows are still a very active area of research.

[2]To keep the table short, I have deliberately omitted algorithms whose running time depends on the maximum capacity, the sum of the capacities, or the maximum flow value. Even with this restriction, the table is incomplete!

Technique	Direct	With dynamic trees	Source(s)
Blocking flow	$O(V^2E)$	$O(VE \log V)$	[Dinits; Sleator and Tarjan]
Network simplex	$O(V^2E)$	$O(VE \log V)$	[Dantzig; Goldfarb and Hao; Goldberg, Grigoriadis, and Tarjan]
Push-relabel (generic)	$O(V^2E)$	—	[Goldberg and Tarjan]
Push-relabel (FIFO)	$O(V^3)$	$O(V^2 \log(V^2/E))$	[Goldberg and Tarjan]
Push-relabel (highest label)	$O(V^2\sqrt{E})$	—	[Cheriyan and Maheshwari; Tunçel]
Pseudoflow	$O(V^2E)$	$O(VE \log V)$	[Hochbaum]
Pseudoflow (highest label)	$O(V^3)$	$O(VE \log(V^2/E))$	[Hochbaum and Orlin]
Incremental BFS	$O(V^2E)$	$O(VE \log(V^2/E))$	[Goldberg, Hed, Kaplan, Tarjan, and Werneck]
Compact abundance graphs		$O(VE)$	[Orlin]

Several purely combinatorial maximum-flow algorithms and their running times.

The fastest maximum flow algorithm known, announced by James Orlin in 2012, runs in ***O(VE) time***, exactly matching the worst-case complexity of a flow decomposition. The details of Orlin's algorithm are far beyond the scope of this course; in addition to his own new techniques, Orlin uses several older algorithms and data structures as black boxes, most of which are themselves quite complicated. (In particular, orlin's algorithm does *not* construct an explicit flow decomposition; in fact, for graphs with only $O(V)$ edges, an extension of his algorithm actually runs in only $O(V^2/\log V)$ time!) Nevertheless, for purposes of analyzing algorithms that *use* maximum flows, this is the time bound you should cite. So write the following sentence on your cheat sheets and cite it in your homeworks:

Maximum flows can be computed in O(VE) time.

Exercises

1. Suppose you are given a directed graph $G = (V, E)$, two vertices s and t, a capacity function $c : E \to \mathbb{R}^+$, and a second function $f : E \to \mathbb{R}$. Describe an algorithm to determine whether f is a maximum (s, t)-flow in G.

2. Let (S, T) and (S', T') be minimum (s, t)-cuts in some flow network G. Prove that $(S \cap S', T \cup T')$ and $(S \cup S', T \cap T')$ are also minimum (s, t)-cuts in G.

3. (a) Describe an efficient algorithm to determine whether a given flow network contains a *unique* maximum (s, t)-flow.
   (b) Describe an efficient algorithm to determine whether a given flow network contains a *unique* minimum (s, t)-cut.

4. Fix any flow network $G = (V, E)$. Our observation that any weighted sum of (s, t)-flows is itself an (s, t)-flow implies that the set of all (s, t)-flows in any graph actually define a *vector space* over the reals.
   (a) Prove that the dimension of this vector space is exactly $E - V + 2$.
   (b) Let T be any spanning tree of G. Prove that the following collection of paths and cycles define a basis for this vector space:

- The unique path in T from s to t;
- The unique cycle in $T \cup \{e\}$, for every edge $e \notin T$.

(c) Let T be any spanning tree of G, and let F be the forest obtained by deleting any single edge in T. Prove that the following collection of paths and cycles define a basis for this vector space:

- The unique path in $F \cup \{e\}$ from s to t, for every edge $e \notin F$ that has one endpoint in each component of F;
- The unique cycle in $F \cup \{e\}$, for every edge $e \notin F$ with both endpoints in the same component of F.

5. Cuts are sometimes defined as subsets of the edges of the graph, instead of as partitions of its vertices. In this problem, you will prove that these two definitions are *almost* equivalent.

We say that a subset X of (directed) edges *separates s and t* if every directed path from s to t contains at least one (directed) edge in X. For any subset S of *vertices*, let $\boldsymbol{\delta S}$ denote the set of directed edges leaving S; that is, $\delta S := \{u \to v \mid u \in S, v \notin S\}$.

(a) Prove that if (S, T) is an (s, t)-cut, then δS separates s and t.

(b) Let X be an arbitrary subset of edges that separates s and t. Prove that there is an (s, t)-cut (S, T) such that $\delta S \subseteq X$.

(c) Let X be a *minimal* subset of edges that separates s and t. (Such a set of edges is sometimes called a ***bond***.) Prove that there is an (s, t)-cut (S, T) such that $\delta S = X$.

6. Suppose instead of capacities, we consider networks where each edge $u \to v$ has a non-negative ***demand*** $d(u \to v)$. Now an (s, t)-flow f is *feasible* if and only if $f(u \to v) \geq d(u \to v)$ for every edge $u \to v$. (Feasible flow values can now be arbitrarily large.) A natural problem in this setting is to find a feasible (s, t)-flow of *minimum* value.

(a) Describe an efficient algorithm to compute a feasible (s, t)-flow, given the graph, the demand function, and the vertices s and t as input. *[Hint: Find a flow that is non-zero everywhere, and then scale it up to make it feasible.]*

(b) Suppose you have access to a subroutine MaxFlow that computes *maximum* flows in networks with edge capacities. Describe an efficient algorithm to compute a *minimum* flow in a given network with edge demands; your algorithm should call MaxFlow exactly once.

(c) State and prove an analogue of the max-flow min-cut theorem for this setting. (Do minimum flows correspond to maximum cuts?)

7. For any flow network G and any vertices u and v, let $bottleneck_G(u, v)$ denote the maximum, over all paths π in G from u to v, of the minimum-capacity edge along π.

(a) Describe and analyze an algorithm to compute $bottleneck_G(s, t)$ in $O(E \log V)$ time.

(b) Describe an algorithm to construct a spanning tree T of G such that $bottleneck_T(u, v) = bottleneck_G(u, v)$ for *all* vertices u and v. (Edges in T inherit their capacities from G.)

8. Suppose you are given a flow network G with ***integer*** edge capacities and an ***integer*** maximum flow f^* in G. Describe algorithms for the following operations:

   (a) INCREMENT(e): Increase the capacity of edge e by 1 and update the maximum flow.

   (b) DECREMENT(e): Decrease the capacity of edge e by 1 and update the maximum flow.

   Both algorithms should modify f^* so that it is still a maximum flow, more quickly than recomputing a maximum flow from scratch.

9. Let G be a network with integer edge capacities. An edge in G is *upper-binding* if increasing its capacity by 1 also increases the value of the maximum flow in G. Similarly, an edge is *lower-binding* if decreasing its capacity by 1 also decreases the value of the maximum flow in G.

   (a) Does every network G have at least one upper-binding edge? Prove your answer is correct.

   (b) Does every network G have at least one lower-binding edge? Prove your answer is correct.

   (c) Describe an algorithm to find all upper-binding edges in G, given both G and a maximum flow in G as input, in $O(E)$ time.

   (d) Describe an algorithm to find all lower-binding edges in G, given both G and a maximum flow in G as input, in $O(EV)$ time.

10. A new assistant professor, teaching maximum flows for the first time, suggests the following greedy modification to the generic Ford-Fulkerson augmenting path algorithm. Instead of maintaining a residual graph, just reduce the capacity of edges along the augmenting path! In particular, whenever we saturate an edge, just remove it from the graph.

```
GreedyFlow(G, c, s, t):
  for every edge e in G
      f(e) ← 0

  while there is a path from s to t
      π ← an arbitrary path from s to t
      F ← minimum capacity of any edge in π
      for every edge e in π
          f(e) ← f(e) + F
          if c(e) = F
              remove e from G
          else
              c(e) ← c(e) − F

  return f
```

   (a) Show that GREEDYFLOW does not always compute a maximum flow.

   (b) Show that GREEDYFLOW is not even guaranteed to compute a good approximation to the maximum flow. That is, for any constant $\alpha > 1$, there is a flow network G such that the value of the maximum flow is more than α times the value of the flow computed by GREEDYFLOW. *[Hint: Assume that GREEDYFLOW chooses the worst possible path π at each iteration.]*

11. A given flow network G may have more than one minimum (s,t)-cut. Let's define the ***best*** minimum (s,t)-cut to be any minimum cut (S,T) with the smallest number of edges crossing from S to T.

    (a) Describe an efficient algorithm to find the best minimum (s,t)-cut when the capacities are integers.

    (b) Describe an efficient algorithm to find the best minimum (s,t)-cut for *arbitrary* edge capacities.

    (c) Describe an efficient algorithm to determine whether a given flow network contains a unique *best* minimum (s,t)-cut.

12. We can speed up the Edmonds-Karp "fat pipe" heuristic, at least for integer capacities, by relaxing our requirements for the next augmenting path. Instead of finding the augmenting path with maximum bottleneck capacity, we find a path whose bottleneck capacity is at least half of maximum, using the following ***capacity scaling*** algorithm.

    The algorithm maintains a bottleneck threshold Δ; initially, Δ is the maximum capacity among all edges in the graph. In each *phase*, the algorithm augments along paths from s to t in which every edge has residual capacity at least Δ. When there is no such path, the phase ends, we set $\Delta \leftarrow \lfloor \Delta/2 \rfloor$, and the next phase begins.

    (a) How many phases will the algorithm execute in the worst case, if the edge capacities are integers?

    (b) Let f be the flow at the end of a phase for a particular value of Δ. Prove that the capacity of a minimum cut in the residual graph G_f is at most $E \cdot \Delta$.

    (c) Prove that in each phase of the scaling algorithm, there are at most $2E$ augmentations.

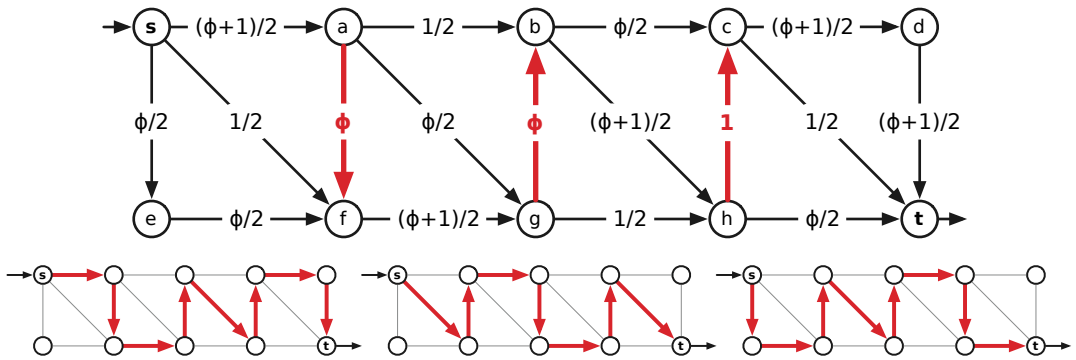
    (d) What is the overall running time of the scaling algorithm, assuming all the edge capacities are integers?

13. An ***(s,t)-series-parallel*** graph is an directed acyclic graph with two designated vertices s (the *source*) and t (the *target* or *sink*) and with one of the following structures:

    - **Base case:** A single directed edge from s to t.
    - **Series:** The union of an (s,u)-series-parallel graph and a (u,t)-series-parallel graph that share a common vertex u but no other vertices or edges.
    - **Parallel:** The union of two smaller (s,t)-series-parallel graphs with the same source s and target t, but with no other vertices or edges in common.

    Describe an efficient algorithm to compute a maximum flow from s to t in an (s,t)-series-parallel graph with arbitrary edge capacities.

14. In 1980 Maurice Queyranne published the following example of a flow network where Edmonds and Karp's "fat pipe" heuristic does not halt. Here, as in Zwick's bad example for the original Ford-Fulkerson algorithm, ϕ denotes the inverse golden ratio $(\sqrt{5}-1)/2$. The three vertical edges play essentially the same role as the horizontal edges in Zwick's example.



Queyranne's network, and a sequence of "fat-pipe" augmentations.

(a) Show that the following infinite sequence of path augmentations is a valid execution of the Edmonds-Karp "fat pipe" algorithm. (See the figure above.)

```
QueyranneFatPipes:
  for i ← 1 to ∞
      push ϕ^(3i−2) units of flow along  s→a→f→g→b→h→c→d→t
      push ϕ^(3i−1) units of flow along  s→f→a→b→g→h→c→t
      push ϕ^(3i)   units of flow along  s→e→f→a→g→b→c→h→t
  forever
```

(b) Describe a sequence of $O(1)$ path augmentations that yields a maximum flow in Queyranne's network.