Unless a problem specifically states otherwise, you may assume a function \textsc{random} that takes a positive integer \( k \) as input and returns an integer chosen uniformly and independently at random from \( \{1, 2, \ldots, k\} \) in \( O(1) \) time. For example, to flip a fair coin, you could call \textsc{random}(2).

1. Suppose we are given a two-dimensional array \( M[1..n, 1..n] \) in which every row and every column is sorted in increasing order and no two elements are equal.

   (a) Describe and analyze an algorithm to solve the following problem in \( O(n) \) time: Given indices \( i, j, i', j' \) as input, compute the number of elements of \( M \) larger than \( M[i, j] \) and smaller than \( M[i', j'] \).

   (b) Describe and analyze an algorithm to solve the following problem in \( O(n) \) time: Given indices \( i, j, i', j' \) as input, return an element of \( M \) chosen uniformly at random from the elements larger than \( M[i, j] \) and smaller than \( M[i', j'] \). Assume the requested range is always non-empty.

   (c) Describe and analyze a randomized algorithm to compute the median element of \( M \) in \( O(n \log n) \) expected time.

2. **Tabulated hashing** uses tables of random numbers to compute hash values. Suppose \(|U| = 2^w \times 2^w \) and \( m = 2^\ell \), so the items being hashed are pairs of \( w \)-bit strings (or \( 2w \)-bit strings broken in half) and hash values are \( \ell \)-bit strings.

   Let \( A[0..2^w-1] \) and \( B[0..2^w-1] \) be arrays of independent random \( \ell \)-bit strings, and define the hash function \( h_{A,B} : U \to [m] \) by setting
   \[
   h_{A,B}(x, y) := A[x] \oplus B[y]
   \]
   where \( \oplus \) denotes bit-wise exclusive-or. Let \( \mathcal{H} \) denote the set of all possible functions \( h_{A,B} \). Filling the arrays \( A \) and \( B \) with independent random bits is equivalent to choosing a hash function \( h_{A,B} \in \mathcal{H} \) uniformly at random.

   (a) Prove that \( \mathcal{H} \) is 2-uniform.

   (b) Prove that \( \mathcal{H} \) is 3-uniform. \([\text{Hint: Solve part (a) first.}]\)

   (c) Prove that \( \mathcal{H} \) is \textbf{not} 4-uniform.

Yes, “see part (b)” is worth full credit for part (a), but only if your solution to part (b) is correct.