OLD CS 473: Fundamental Algorithms, Spring 2015

Discussion 14

April 30, 2015

14.1. NP Completeness.

Show that the following problems are NP-COMPLETE.

Max Degree Spanning Tree

Instance: Graph $G = (V, E)$ and integer $k$

Question: Does $G$ contain a spanning tree $T$ where every node in $T$ is of degree at most $k$?

TILING

Instance: Finite set $\mathcal{RECT}$ of rectangles and a rectangle $R$ in the plane.

Question: Is there a way of placing the rectangles of $\mathcal{RECT}$ inside $R$, so that no pair of the rectangles intersect, and all the rectangles have their edges parallel of the edges of $R$?

HITTING SET

Instance: A collection $C$ of subsets of a set $S$, a positive integer $K$.

Question: Does $S$ contain a hitting set for $C$ of size $K$ or less, that is, a subset $S' \subseteq S$ with $|S'| \leq K$ and such that $S'$ contains at least one element from each subset in $C$.

LARGEST COMMON SUBGRAPH

Instance: Graphs $G = (V_1, E_1), H = (V_2, E_2)$, positive integer $K$.

Question: Do there exists subsets $E'_1 \subseteq E_1$ and $E'_2 \subseteq E_2$ with $|E'_1| = |E'_2| \geq K$ such that the two subgraphs $G' = (V_1, E'_1)$ and $H' = (V_2, E'_2)$ are isomorphic?

BIN PACKING

Instance: Finite set $U$ of items, a size $s(u) \in \mathbb{Z}^+$ for each $u \in U$, an integer bin capacity $B$, and a positive integer $K$.

Question: Is there a partition of $U$ int disjoint sets $U_1, \ldots, U_K$ such that the sum of the sizes of the items inside each $U_i$ is $B$ or less?

14.2. Self reducibility!

For each of the following problems, assume you are given a black box that can solve the decision problem in polynomial time. Show how to solve the optimization version of this problem in polynomial time using this black box.
Shortest Path

**Instance:** A weighted undirected graph $G$, vertices $s$ and $t$ and a threshold $w$.

**Question:** Is there a path between $s$ and $t$ in $G$ of length at most $w$?

Independent Set

**Instance:** A graph $G$, integer $k$.

**Question:** Is there an independent set in $G$ of size $k$?

3Colorable

**Instance:** A graph $G$.

**Question:** Is there a coloring of $G$ using three colors?

TSP

**Instance:** A weighted undirected graph $G$, and a threshold $w$.

**Question:** Is there a TSP tour of $G$ of weight at most $w$?

Vertex Cover

**Instance:** A graph $G$, integer $k$.

**Question:** Is there a vertex cover in $G$ of size $k$?

Subset Sum

**Instance:** $S$ - set of positive integers, $t$: - an integer number (target).

**Question:** Is there a subset $X \subseteq S$ such that $\sum_{x \in X} x = t$?

3DM

**Instance:** $X, Y, Z$ sets of $n$ elements, and $T$ a set of triples, such that $(a, b, c) \in T \subseteq X \times Y \times Z$.

**Question:** Is there a subset $S \subseteq T$ of $n$ disjoint triples, s.t. every element of $X \cup Y \cup Z$ is covered exactly once.?

Partition

**Instance:** A set $S$ of $n$ numbers.

**Question:** Is there a subset $T \subseteq S$ s.t. $\sum_{t \in T} t = \sum_{s \in S \setminus T} s$?

SET COVER

**Instance:** $(X, \mathcal{F}, k)$:

- $X$: A set of $n$ elements
- $\mathcal{F}$: A family of subsets of $S$, s.t. $\bigcup_{X \in \mathcal{F}} X = X$.
- $k$: A positive integer.

**Question:** Are there $k$ sets $S_1, \ldots, S_k \in \mathcal{F}$ that cover $S$. Formally, $\bigcup_{i} S_i = X$?
**CYCLE HATER.**

**Instance:** An undirected graph $G = (V, E)$, and an integer $k > 0$.

**Question:** Is there a subset $X \subseteq V$ of at most $k$ vertices, such that all cycles in $G$ contain at least one vertices of $X$.

**CYCLE LOVER.**

**Instance:** An undirected graph $G = (V, E)$, and an integer $k > 0$.

**Question:** Is there a subset $X \subseteq V$ of at most $k$ vertices, such that all cycles in $G$ contain at least two vertices of $X$.

### 14.3. INDEPENDENCE

Let $G = (V, E)$ be an undirected graph over $n$ vertices. Assume that you are given a numbering $\pi : V \to \{1, \ldots, n\}$ (i.e., every vertex have a unique number), such that for any edge $ij \in E$, we have $|\pi(i) - \pi(j)| \leq 20$.

Either prove that it is $NP$-Hard to find the largest independent set in $G$, or provide a polynomial time algorithm.