Chapter 26

Approximation Algorithms using Linear Programming

OLD CS 473: Fundamental Algorithms, Spring 2015
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26.0.1 Weighted vertex cover

26.0.2 Weighted vertex cover

26.0.2.1 Weighted vertex cover

Weighted Vertex Cover problem $G = (V, E)$.
Each vertex $v \in V$: cost $c_v$.
Compute a vertex cover of minimum cost.

(A) vertex cover: subset of vertices $V$ so each edge is covered.
(B) NP-Hard
(C) ... unweighted Vertex Cover problem.
(D) ... write as an integer program (IP):
(E) $\forall v \in V: x_v = 1 \iff v$ in the vertex cover.
(F) $\forall vu \in E$: covered. $\implies x_v \lor x_u$ true. $\implies x_v + x_u \geq 1$.
(G) minimize total cost: $\min \sum_{v \in V} x_v c_v$.

26.0.3 Weighted vertex cover

26.0.3.1 State as IP $\implies$ Relax $\implies$ LP

$$\min \sum_{v \in V} c_v x_v,$$

such that $x_v \in \{0, 1\}$ \quad $\forall v \in V$

$x_v + x_u \geq 1$ \quad $\forall vu \in E$. 

(26.1)
26.0.3.2 Weighted vertex cover – rounding the LP

(A) Optimal solution to this LP: $\hat{\alpha}$ value of var $X_v$, $\forall v \in V$.
(B) optimal value of LP solution is $\hat{\alpha} = \sum_{v \in V} c_v \hat{x}_v$.
(C) optimal integer solution: $x^I_v$, $\forall v \in V$ and $\alpha^I$.
(D) Any valid solution to IP is valid solution for LP!
(E) $\hat{\alpha} \leq \alpha^I$.

Integral solution not better than LP.

(F) Got fractional solution (i.e., values of $\hat{x}_v$).
(G) Fractional solution is better than the optimal cost.

(H) Q: How to turn fractional solution into a (valid!) integer solution?

(I) Using rounding.

26.0.3.3 How to round?

(A) consider vertex $v$ and fractional value $\hat{x}_v$.
(B) If $\hat{x}_v = 1$ then include in solution!
(C) If $\hat{x}_v = 0$ then do not include in solution.
(D) if $\hat{x}_v = 0.9$ $\implies$ LP considers $v$ as being 0.9 useful.
(E) The LP puts its money where its belief is...
(F) $\hat{\alpha}$ value is a function of this “belief” generated by the LP.

(G) Big idea: Trust LP values as guidance to usefulness of vertices.

26.0.3.4 II: How to round?

(A) Pick all vertices $\geq$ threshold of usefulness according to LP.

(B) $S = \{ v \mid \hat{x}_v \geq 1/2 \}$.

(C) Claim: $S$ a valid vertex cover, and cost is low.

(min) $\sum_{v \in V} c_v x_v,$

s.t. $0 \leq x_v \quad \forall v \in V,$

$x_v \leq 1 \quad \forall v \in V,$

$x_v + x_u \geq 1 \quad \forall vu \in E.$

(A) Indeed, edge cover as: $\forall vu \in E$ have $\hat{x}_v + \hat{x}_u \geq 1.$

(B) $\hat{x}_v, \hat{x}_u \in (0, 1)$

$\implies \hat{x}_v \geq 1/2$ or $\hat{x}_u \geq 1/2.$

$\implies v \in S$ or $u \in S$ (or both).

$\implies S$ covers all the edges of $G.$
26.0.3.5 Cost of solution

Cost of $S$:

$$c_S = \sum_{v \in S} c_v = \sum_{v \in S} 1 \cdot c_v \leq \sum_{v \in S} 2\hat{c}_v \cdot c_v \leq 2 \sum_{v \in V} \hat{c}_v c_v = 2\alpha' \leq 2\alpha,$$

since $\hat{c}_v \geq 1/2$ as $v \in S$.

$\alpha'$ is cost of the optimal solution $\implies$

**Theorem 26.0.1.** The **Weighted Vertex Cover** problem can be 2-approximated by solving a single LP. Assuming computing the LP takes polynomial time, the resulting approximation algorithm takes polynomial time.

26.0.4 The lessons we can take away

26.0.4.1 Or not - boring, boring, boring.

(A) Weighted vertex cover is simple, but resulting approximation algorithm is non-trivial.
(B) Not aware of any other 2-approximation algorithm does not use LP. (For the weighted case!)
(C) Solving a relaxation of an optimization problem into a LP provides us with insight.
(D) But... have to be creative in the rounding.

26.0.5 Revisiting Set Cover

26.0.5.1 Revisiting Set Cover

(A) Purpose: See new technique for an approximation algorithm.
(B) Not better than greedy algorithm already seen $O(\log n)$ approximation.

**Problem:** **Set Cover**

<table>
<thead>
<tr>
<th><strong>Instance:</strong> $(S, F)$</th>
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<tbody>
<tr>
<td>$S$ - a set of $n$ elements</td>
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**Question:** The set $\mathcal{X} \subseteq F$ such that $\mathcal{X}$ contains as few sets as possible, and $\mathcal{X}$ covers $S$.

26.0.5.2 **Set Cover – IP & LP**

$$\begin{align*}
\min & \quad \alpha = \sum_{U \in \mathcal{F}} x_U, \\
\text{s.t.} & \quad x_U \in \{0, 1\} \quad \forall U \in \mathcal{F}, \\
& \quad \sum_{U \in \mathcal{F}, s \in U} x_U \geq 1 \quad \forall s \in S.
\end{align*}$$
Next, we relax this IP into the following LP.

\[
\begin{align*}
\min & \quad \alpha = \sum_{U \in \mathcal{F}} x_U, \\
0 & \leq x_U \leq 1 \quad \forall U \in \mathcal{F}, \\
\sum_{U \in \mathcal{F}, s \in U} x_U & \geq 1 \quad \forall s \in S.
\end{align*}
\]

26.0.5.3 **Set Cover – IP & LP**

(A) **LP solution**: \( \forall U \in \mathcal{F}, x_U^{\text{LP}}, \) and \( \hat{\alpha}. \)

(B) **Opt IP solution**: \( \forall U \in \mathcal{F}, x_U^{\text{IP}}, \) and \( \alpha^{\text{IP}}. \)

(C) Use LP solution to guide in rounding process.

(D) If \( x_U^{\text{LP}} \) is close to 1 then pick \( U \) to cover.

(E) If \( x_U^{\text{LP}} \) close to 0 do not.

(F) **Idea**: Pick \( U \in \mathcal{F}: \) randomly choose \( U \) with **probability** \( x_U^{\text{LP}}. \)

(G) Resulting family of sets \( \mathcal{G}. \)

(H) \( Z_S: \) indicator variable. 1 if \( S \in \mathcal{G}. \)

(I) Cost of \( \mathcal{G} \) is \( \sum_{S \in \mathcal{G}} Z_S, \) and the expected cost is \( E[\text{cost of } \mathcal{G}] = E[\sum_{S \in \mathcal{G}} Z_S] = \sum_{S \in \mathcal{G}} E[Z_S] = \sum_{S \in \mathcal{G}} \Pr[S \in \mathcal{G}] = \sum_{S \in \mathcal{G}} \hat{x}_S = \hat{\alpha} \leq \alpha^{\text{IP}}. \)

(J) In expectation, \( \mathcal{G} \) is not too expensive.

(K) Bigus problemos: \( \mathcal{G} \) might fail to cover some element \( s \in S. \)

26.0.5.4 **Set Cover – Rounding continued**

(A) **Solution**: Repeat rounding stage \( m = 10\lceil \log n \rceil = O(\log n) \) times.

(B) \( n = |S|. \)

(C) \( \mathcal{G}_i: \) random cover computed in \( i \)th iteration.

(D) \( \mathcal{H} = \cup_i \mathcal{G}_i. \) Return \( \mathcal{H} \) as the required cover.

26.0.5.5 **The set \( \mathcal{H} \) covers \( S \)**

(A) For an element \( s \in S, \) we have that

\[
\sum_{U \in \mathcal{F}, s \in U} \hat{x}_U \geq 1, 
\]

(26.2)

(B) probability \( s \) not covered by \( \mathcal{G}_i \) (ith iteration set).

\[
\Pr[s \text{ not covered by } \mathcal{G}_i] = \Pr[\text{ no } U \in \mathcal{F}, \text{ s.t. } s \in U \text{ picked into } \mathcal{G}_i] = \prod_{U \in \mathcal{F}, s \in U} \Pr[U \text{ was not picked into } \mathcal{G}_i]
\]

\[
= \prod_{U \in \mathcal{F}, s \in U} (1 - \hat{x}_U) \leq \prod_{U \in \mathcal{F}, s \in U} \exp(-\hat{x}_U)
\]

\[
= \exp\left(-\sum_{U \in \mathcal{F}, s \in U} \hat{x}_U\right) \leq \exp(-1) \leq \frac{1}{2}, \leq \frac{1}{2}
\]
26.0.6 The set $\mathcal{H}$ covers $S$

26.0.6.1 Probability of a single item to be covered

(A) $\Pr[s \text{ not covered by } G_1] \leq 1/2$.
(B) Number of iterations of rounding $m = O(\log n)$.
(C) Covering with sets in $G_1, \ldots, G_m$.
(D) Probability $s$ is not covered in all $m$ iterations

$$P_s = \Pr[s \text{ not covered by } G_1, \ldots, G_m]$$

$$\leq \Pr[(s \notin F_1) \cap (s \notin F_2) \cap \ldots \cap (s \notin F_m)]$$

$$\leq \Pr[s \notin F_1] \Pr[s \notin F_2] \cdots \Pr[s \notin F_m]$$

$$= \frac{1}{2} \times \frac{1}{2} \times \cdots \times \frac{1}{2} = \left(\frac{1}{2}\right)^m < \frac{1}{n^{10}}.$$

26.0.7 The set $\mathcal{H}$ covers $S$

26.0.7.1 Probability of all items to be covered

(A) $n = |S|$,
(B) Probability of $s \in S$, not to be in $G_1 \cup \ldots \cup F_m$ is

$$P_s < \frac{1}{n^{10}}.$$

(C) Probability one of $n$ elements of $S$ is not covered by $\mathcal{H}$ is

$$\sum_{s \in S} \Pr[s \notin G_1 \cup \ldots \cup F_m] = \sum_{s \in S} P_s < n(1/n^{10}) = 1/n^9.$$

XXX

26.0.7.2 Reminder: LP for Set Cover

$$\min \quad \alpha = \sum_{U \in \mathcal{F}} x_U,$$

$$0 \leq x_U \leq 1 \quad \forall U \in \mathcal{F},$$

$$\sum_{U \in \mathcal{F}, s \in U} x_U \geq 1 \quad \forall s \in S.$$

(A) Solve the LP.
(B) $\hat{x}_U$: Value of $x_u$ in the optimal LP solution.
(C) Fractional solution: $\hat{\alpha} = \sum_{U \in \mathcal{F}} \hat{x}_U$.
(D) Integral solution (what we want): $\alpha^I \geq \hat{\alpha}$.  

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26.0.7.3 Cost of solution

(A) \((S, F)\): Given instance of \textbf{Set Cover}.
(B) For \(U \in F\), \(\tilde{x}_U\): LP value for set \(U\) in optimal solution.
(C) For \(\mathcal{S}_i\): Indicator variable \(Z_u = 1 \iff U \in \mathcal{S}_i\).
(D) Expected number of sets in the \(i\)th sample:
\[
E[|\mathcal{S}_i|] = E\left[\sum_{U \in \mathcal{F}} Z_U\right] = \sum_{U \in \mathcal{F}} E[Z_U] = \sum_{U \in \mathcal{F}} \tilde{x}_U = \hat{\alpha} \leq \alpha^I.
\]
(E) \(\implies\) Each iteration expected cost of cover \(\leq\) cost of optimal solution (i.e., \(\alpha^I\)). XXX
(F) Expected size of the solution is
\[
E[|\mathcal{M}|] = E[|\cup_i \mathcal{S}_i|] \leq E\left[\sum_i |\mathcal{S}_i|\right] \leq m\alpha^I = O(\alpha^I \log n).
\]

26.0.7.4 The result

\textbf{Theorem 26.0.2.} By solving an LP one can get an \(O(\log n)\)-approximation to \textbf{Set Cover} by a randomized algorithm. The algorithm succeeds with high probability.

26.0.8 Minimizing congestion

26.0.8.1 Minimizing congestion by example

26.0.8.2 Minimizing congestion

(A) \(G\): graph. \(n\) vertices.
(B) \(\pi_i, \sigma_i\) paths with the same endpoints \(v_i, u_i \in V(G)\), for \(i = 1, \ldots, t\).
(C) Rule I: Send one unit of flow from \(v_i\) to \(u_i\).
(D) Rule II: Choose whether to use \(\pi_i\) or \(\sigma_i\).
(E) Target: No edge in \(G\) is being used too much.

\textbf{Definition 26.0.3.} Given a set \(X\) of paths in a graph \(G\), the \textbf{congestion} of \(X\) is the maximum number of paths in \(X\) that use the same edge.
26.0.8.3 Minimizing congestion

(A) **IP $\implies$ LP:**

$$\begin{align*}
\min & \quad w \\
\text{s.t.} & \quad x_i \geq 0 & i = 1, \ldots, t, \\
& \quad x_i \leq 1 & i = 1, \ldots, t, \\
& \quad \sum_{e \in \pi_i} x_i + \sum_{e \in \sigma_i} (1 - x_i) \leq w & \forall e \in E.
\end{align*}$$

(B) $\hat{x}_i$: value of $x_i$ in the optimal LP solution.

(C) $\hat{w}$: value of $w$ in LP solution.

(D) Optimal congestion must be bigger than $\hat{w}$.

(E) $X_i$: random variable one with probability $\hat{x}_i$, and zero otherwise.

(F) If $X_i = 1$ then use $\pi$ to route from $v_i$ to $u_i$.

(G) Otherwise use $\sigma_i$.

26.0.8.4 Minimizing congestion

(A) Congestion of $e$ is $Y_e = \sum_{e \in \pi_i} X_i + \sum_{e \in \sigma_i} (1 - X_i)$.

(B) And in expectation

$$\alpha_e = E[Y_e] = E\left[\sum_{e \in \pi_i} X_i + \sum_{e \in \sigma_i} (1 - X_i)\right]$$

$$= \sum_{e \in \pi_i} E[X_i] + \sum_{e \in \sigma_i} E[(1 - X_i)]$$

$$= \sum_{e \in \pi_i} \hat{x}_i + \sum_{e \in \sigma_i} (1 - \hat{x}_i) \leq \hat{w}.$$

(C) $\hat{w}$: Fractional congestion (from LP solution).

26.0.8.5 Minimizing congestion - continued

(A) $Y_e = \sum_{e \in \pi_i} X_i + \sum_{e \in \sigma_i} (1 - X_i)$.

(B) $Y_e$ is just a sum of independent 0/1 random variables!

(C) Chernoff inequality tells us sum can not be too far from expectation!

26.0.8.6 Minimizing congestion - continued

(A) By Chernoff inequality:

$$\Pr[Y_e \geq (1 + \delta)\alpha_e] \leq \exp\left(-\frac{\alpha_e\delta^2}{4}\right) \leq \exp\left(-\frac{\hat{w}\delta^2}{4}\right).$$
(B) Let $\delta = \sqrt{\frac{400}{\tilde{w}}} \ln t$. We have that

\[ \Pr \left[ Y_e \geq (1 + \delta)\alpha_e \right] \leq \exp \left( -\frac{\delta^2 \tilde{w}}{4} \right) \leq \frac{1}{t^{100}}, \]

(C) If $t \geq n^{1/50} \implies \forall$ edges in graph congestion $\leq (1 + \delta)\tilde{w}$.

(D) $t$: Number of pairs, $n$: Number of vertices in $G$.

26.0.8.7 Minimizing congestion - continued

(A) Got: For $\delta = \sqrt{\frac{400}{\tilde{w}} \ln t}$. We have

\[ \Pr \left[ Y_e \geq (1 + \delta)\alpha_e \right] \leq \exp \left( -\frac{\delta^2 \tilde{w}}{4} \right) \leq \frac{1}{t^{100}}, \]

(B) Play with the numbers. If $t = n$, and $\tilde{w} \geq \sqrt{n}$. Then, the solution has congestion larger than the optimal solution by a factor of

\[ 1 + \delta = 1 + \sqrt{\frac{20}{\tilde{w}}} \ln t \leq 1 + \frac{\sqrt{20 \ln n}}{n^{1/4}}, \]

which is of course extremely close to 1, if $n$ is sufficiently large.

26.0.8.8 Minimizing congestion: result

Theorem 26.0.4. (A) $G$: Graph $n$ vertices.

(B) $(s_1, t_1), \ldots, (s_t, t_t)$: pairs of vertices

(C) $\pi_i, \sigma_i$: two different paths connecting $s_i$ to $t_i$

(D) $\tilde{w}$: Fractional congestion at least $n^{1/2}$.

(E) $\text{opt}$: Congestion of optimal solution.

(F) $\implies$ In polynomial time (LP solving time) choose paths

(A) congestion $\forall$ edges: $\leq (1 + \delta)\text{opt}$

(B) $\delta = \sqrt{\frac{20}{\tilde{w}} \ln t}$.

26.0.8.9 When the congestion is low

(A) Assume $\tilde{w}$ is a constant.

(B) Can get a better bound by using the Chernoff inequality in its more general form.

(C) set $\delta = c \ln t / \ln \ln t$, where $c$ is a constant. For $\mu = \alpha_e$, we have that

\[ \Pr \left[ Y_e \geq (1 + \delta)\mu \right] \leq \left( \frac{e^\delta}{(1 + \delta)^{1+\delta}} \right)^\mu \]

\[ = \exp \left( \mu (\delta - (1 + \delta) \ln(1 + \delta)) \right) \]

\[ = \exp \left( -\mu c' \ln t \right) \leq \frac{1}{t^{O(1)}}, \]
where $c'$ is a constant that depends on $c$ and grows if $c$ grows.

26.0.8.10 When the congestion is low

(A) Just proved that...
(B) if the optimal congestion is $O(1)$, then...
(C) algorithm outputs a solution with congestion $O(\log t / \log \log t)$, and this holds with high probability.

26.0.9 Reminder about Chernoff inequality

26.0.9.1 The Chernoff Bound — General Case

26.0.9.2 Chernoff inequality

Problem 26.0.5. Let $X_1, \ldots, X_n$ be $n$ independent Bernoulli trials, where
\[
\Pr[X_i = 1] = p_i, \quad \Pr[X_i = 0] = 1 - p_i,
\]
\[
Y = \sum_i X_i, \quad \text{and} \quad \mu = \mathbb{E}[Y].
\]
We are interested in bounding the probability that $Y \geq (1 + \delta)\mu$.

26.0.9.3 Chernoff inequality

Theorem 26.0.6 (Chernoff inequality). For any $\delta > 0$,
\[
\Pr[Y > (1 + \delta)\mu] < \left( \frac{e^\delta}{(1 + \delta)^{1+\delta}} \right)^\mu.
\]
Or in a more simplified form, for any $\delta \leq 2e - 1$,
\[
\Pr[Y > (1 + \delta)\mu] < \exp(-\mu\delta^2/4),
\]
and
\[
\Pr[Y > (1 + \delta)\mu] < 2^{-\mu(1+\delta)},
\]
for $\delta \geq 2e - 1$.

26.0.9.4 More Chernoff...

Theorem 26.0.7. Under the same assumptions as the theorem above, we have
\[
\Pr[Y < (1 - \delta)\mu] \leq \exp\left(-\frac{\mu \delta^2}{2}\right).
\]