Approximation Algorithms using Linear Programming

Lecture 26

April 30, 2015

26.1: Weighted vertex cover

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Weighted vertex cover

Weighted Vertex Cover problem

$G = (V, E)$.
Each vertex $\mathbf{v} \in V$: cost $\mathbf{c_v}$.
Compute a vertex cover of minimum cost.

1. vertex cover: subset of vertices V so each edge is covered.
2. NP-Hard
3. ...unweighted Vertex Cover problem.
4. ... write as an integer program (IP):
5. $\forall \mathbf{v} \in V$: $x_v = 1 \iff \mathbf{v}$ in the vertex cover.
6. $\forall \mathbf{vu} \in E$: covered. $\implies x_v \vee x_u$ **true**. $\implies x_v + x_u \geq 1$.
7. minimize total cost: $\min \sum_{v \in V} x_v c_v$.

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State as IP $\implies$ Relax $\implies$ LP

$$\begin{aligned} \min \quad & \sum_{v \in V} c_v x_v, \\ \text{such that} \quad & x_v \in \{0,1\} && \forall v \in V \qquad (1) \\ & x_v + x_u \geq 1 && \forall vu \in E. \end{aligned}$$

1. ... NP-Hard.
2. relax the integer program.
3. allow x_v get values $\in [0,1]$.
4. $x_v \in \{0,1\}$ replaced by $0 \leq x_v \leq 1$. The resulting LP is

$$\begin{aligned} \min \quad & \sum_{v \in V} c_v x_v, \\ \text{s.t.} \quad & 0 \leq x_v && \forall v \in V, \\ & x_v \leq 1 && \forall v \in V, \\ & x_v + x_u \geq 1 && \forall vu \in E. \end{aligned}$$

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Weighted vertex cover – rounding the LP

1. Optimal solution to this LP: $\widehat{x}_v$ value of var X_v, $\forall v \in V$.
2. optimal value of LP solution is $\widehat{\alpha} = \sum_{v \in V} c_v \widehat{x}_v$.
3. optimal integer solution: x^I_v, $\forall v \in V$ and α^I.
4. **Any valid solution to IP is valid solution for LP!**
5. $\widehat{\alpha} \leq \alpha^I$.
   Integral solution not better than LP.
6. Got fractional solution (i.e., values of $\widehat{x}_v$).
7. Fractional solution is better than the optimal cost.
8. Q: How to turn fractional solution into a (valid!) integer solution?
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How to round?

1. consider vertex **v** and fractional value $\widehat{x_v}$.
2. If $\widehat{x_v} = 1$ then include in solution!
3. If $\widehat{x_v} = 0$ then do include in solution.
4. if $\widehat{x_v} = 0.9 \implies$ LP considers **v** as being **0.9** useful.
5. The LP puts its money where its belief is...
6. ...$\widehat{\alpha}$ value is a function of this "belief" generated by the LP.
7. Big idea: Trust LP values as guidance to usefulness of vertices.

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$$\text{s.t.} \quad 0 \leq x_v \qquad \forall v \in V$$
$$x_v \leq 1 \qquad \forall v \in V$$
$$x_v + x_u \geq 1 \qquad \forall vu \in E$$

1. Pick all vertices $\geq$ threshold of usefulness according to LP.
2. $S = \left\{ v \;\middle|\; \widehat{x_v} \geq 1/2 \right\}$.
3. **Claim**: S a valid vertex cover, and cost is low.

1. Indeed, edge cover as: $\forall vu \in E$ have $\widehat{x_v} + \widehat{x_u} \geq 1$.
2. $\widehat{x_v}, \widehat{x_u} \in (0, 1)$
   $\implies \widehat{x_v} \geq 1/2$ or $\widehat{x_u} \geq 1/2$.
   $\implies v \in S$ or $u \in S$ (or both).
   $\implies S$ covers all the edges of G.

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II: How to round?

$$\begin{aligned} \min \quad & \sum_{v \in V} c_v x_v, \\ \text{s.t.} \quad & 0 \le x_v & \forall v \in V \\ & x_v \le 1 & \forall v \in V \\ & x_v + x_u \ge 1 & \forall vu \in E \end{aligned}$$

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Cost of S:

$$c_S = \sum_{v \in S} c_v = \sum_{v \in S} 1 \cdot c_v \le \sum_{v \in S} 2\widehat{x}_v \cdot c_v \le 2 \sum_{v \in V} \widehat{x}_v c_v = 2\widehat{\alpha} \le 2\alpha^I,$$

since $\widehat{x}_v \ge 1/2$ as $v \in S$.

α^I is cost of the optimal solution $\implies$

Theorem

The **Weighted Vertex Cover** *problem can be* 2*-approximated by solving a single* LP*. Assuming computing the* LP *takes polynomial time, the resulting approximation algorithm takes polynomial time.*

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The lessons we can take away

Or not - boring, boring, boring.

1. Weighted vertex cover is simple, but resulting approximation algorithm is non-trivial.
2. Not aware of any other 2-approximation algorithm does not use LP. (For the weighted case!)
3. Solving a **relaxation** of an optimization problem into a LP provides us with insight.
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26.1.2: Revisiting Set Cover

Revisiting Set Cover

1. Purpose: See new technique for an approximation algorithm.
2. Not better than greedy algorithm already seen $O(\log n)$ approximation.

Problem: Set Cover

Instance: $(S, \mathcal{F})$
S - a set of n elements
$\mathcal{F}$ - a family of subsets of S, s.t. $\bigcup_{X \in \mathcal{F}} X = S$.
Question: The set $\mathcal{X} \subseteq F$ such that $\mathcal{X}$ contains as few sets as possible, and $\mathcal{X}$ covers S.

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& \sum_{U \in \mathcal{F}, s \in U} x_U \geq 1 & \forall s \in S.
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Next, we relax this IP into the following LP.

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Set Cover – IP & LP

1. LP solution: $\forall U \in \mathcal{F}$, $\widehat{x_U}$, and $\widehat{\alpha}$.
2. Opt IP solution: $\forall U \in \mathcal{F}$, x_U^I, and α^I.
3. Use LP solution to guide in rounding process.
4. If $\widehat{x_U}$ is close to $\mathbf{1}$ then pick U to cover.
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Set Cover – IP & LP

1. LP solution: $\forall U \in \mathcal{F}$, $\widehat{x_U}$, and $\widehat{\alpha}$.
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Set Cover – Rounding continued

1. **Solution**: Repeat rounding stage $m = 10\lceil \lg n \rceil = O(\log n)$ times.
2. $n = |S|$.
3. $\mathcal{G}_i$: random cover computed in ith iteration.
4. $\mathcal{H} = \cup_i \mathcal{G}_i$. Return $\mathcal{H}$ as the required cover.

Set Cover – Rounding continued

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The set $\mathcal{H}$ covers $\mathbf{S}$

1. For an element $s \in S$, we have that

$$\sum_{U \in \mathcal{F}, s \in U} \widehat{x_U} \geq 1, \tag{2}$$

2. probability s not covered by $\mathcal{G}_i$ (ith iteration set).

$\mathbf{Pr}\left[s \text{ not covered by } \mathcal{G}_i\right]$
$= \mathbf{Pr}\left[\text{ no } U \in \mathcal{F}, \text{ s.t. } s \in U \text{ picked into } \mathcal{G}_i\right]$
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$$= \prod_{U \in \mathcal{F}, s \in U} \left(1 - \widehat{x_U}\right) \leq \prod_{U \in \mathcal{F}, s \in U} \exp\left(-\widehat{x_U}\right)$$
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The set $\mathcal{H}$ covers $\mathbf{S}$

Probability of a single item to be covered

1. $\mathbf{Pr}\left[s \text{ not covered by } \mathcal{G}_i\right] \leq 1/2.$
2. Number of iterations of rounding $m = O(\log n)$.
3. Covering with sets in $\mathcal{G}_1, \ldots, \mathcal{G}_m$.
4. probability s is not covered in all m iterations

$$\begin{aligned}
P_s &= \mathbf{Pr}\left[s \text{ not covered by } \mathcal{G}_1, \ldots, \mathcal{F}_m\right] \\
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Probability of all items to be covered

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$$P_s < \frac{1}{n^{10}}.$$

3. probability one of n elements of S is not covered by $\mathcal{H}$ is

$$\sum_{s \in S} \Pr[s \notin \mathcal{G}_1 \cup \ldots \cup \mathcal{F}_m] = \sum_{s \in S} P_s < n(1/n^{10}) = 1/n^9.$$

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The set $\mathcal{H}$ covers $\mathbf{S}$

Probability of all items to be covered

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XXX

Reminder: LP for Set Cover

$$\begin{aligned} \textbf{min} \quad & \alpha = \sum_{U \in \mathcal{F}} x_U, & \\ & 0 \le x_U \le 1 & \forall U \in \mathcal{F}, \\ & \sum_{U \in \mathcal{F}, s \in U} x_U \ge 1 & \forall s \in S. \end{aligned}$$

1. Solve the LP.
2. $\widehat{x_U}$: Value of x_u in the optimal LP solution.
3. Fractional solution: $\widehat{\alpha} = \sum_{U \in \mathcal{F}} \widehat{x_U}$.
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Cost of solution

1. $(S, \mathcal{F})$: Given instance of **Set Cover**.
2. For $U \in \mathcal{F}$, $\widehat{x_U}$: LP value for set U in optimal solution.
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5. $\implies$ Each iteration expected cost of cover $\leq$ cost of optimal solution (i.e., α^I). XXX
6. Expected size of the solution is

$$\mathbf{E}[|\mathcal{H}|] = \mathbf{E}[|\cup_i \mathcal{G}_i|] \leq \mathbf{E}\left[\sum_i |\mathcal{G}_i|\right] \leq m\alpha^I = O(\alpha^I \log n).$$

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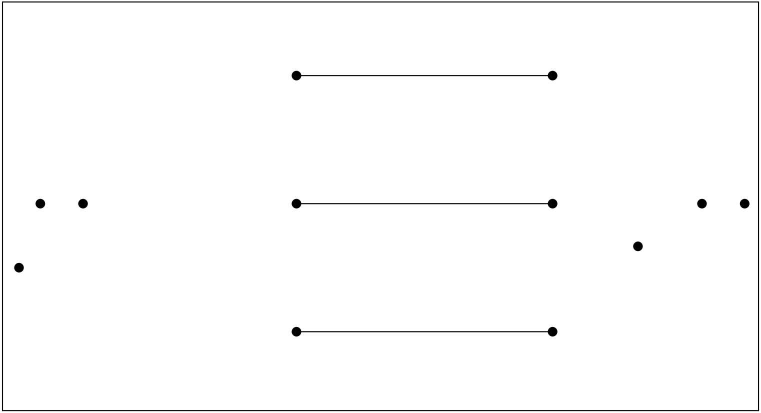
The result

Theorem

By solving an LP *one can get an* $O(\log n)$*-approximation to* **Set Cover** *by a randomized algorithm. The algorithm succeeds with high probability.*

26.1.3: Minimizing congestion

Minimizing congestion by example



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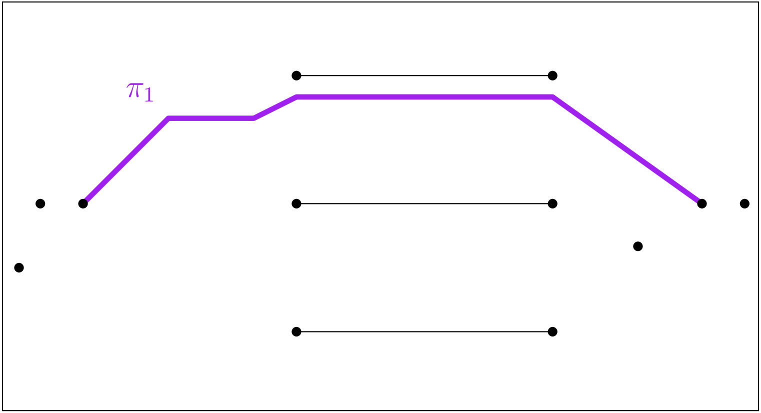

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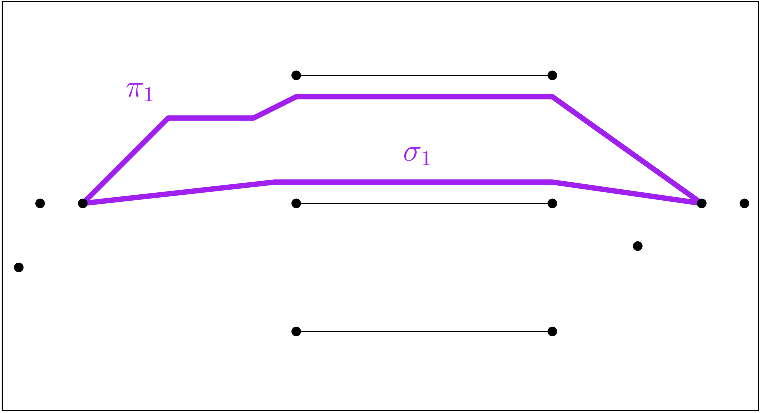

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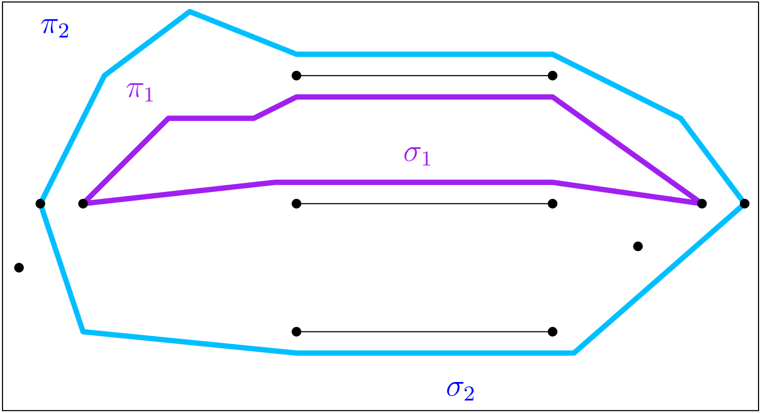

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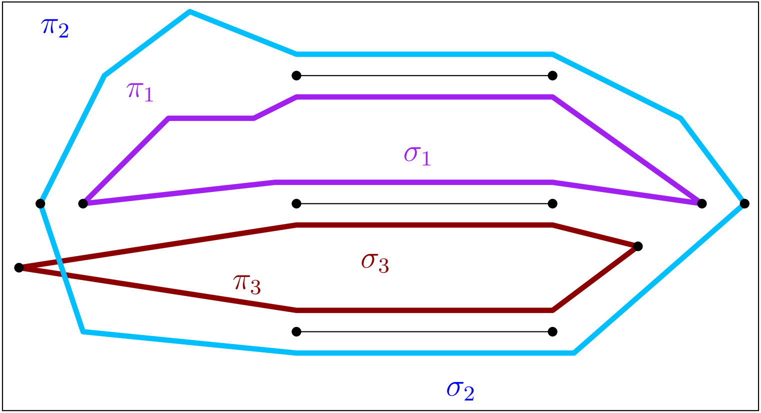

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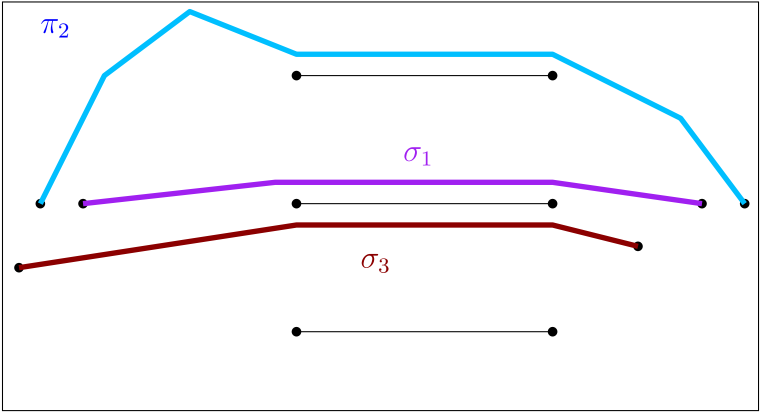

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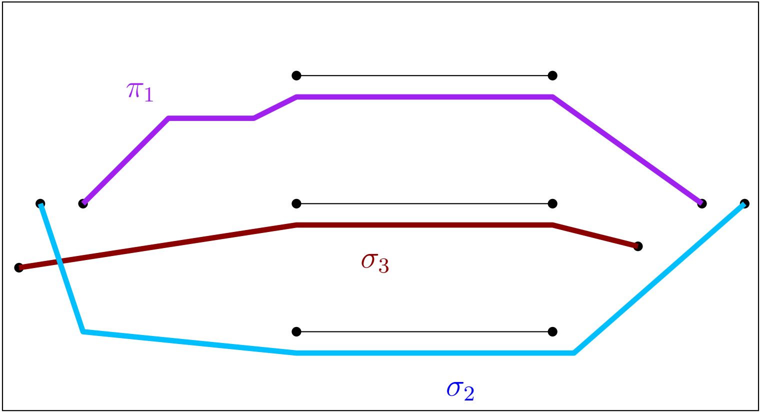

Minimizing congestion

1. G: graph. n vertices.
2. π_i, σ_i paths with the same endpoints $\mathbf{v}_i, \mathbf{u}_i \in V(G)$, for $i = 1, \ldots, t$.
3. Rule I: Send one unit of flow from $\mathbf{v}_i$ to $\mathbf{u}_i$.
4. Rule II: Choose whether to use π_i or σ_i.
5. Target: No edge in G is being used too much.

Definition

Given a set X of paths in a graph G, the **congestion** of X is the maximum number of paths in X that use the same edge.

Minimizing congestion

1. IP $\implies$ LP:

$$\begin{aligned} \min \quad & w \\ \text{s.t.} \quad & x_i \geq 0 & i = 1, \ldots, t, \\ & x_i \leq 1 & i = 1, \ldots, t, \\ & \sum_{e \in \pi_i} x_i + \sum_{e \in \sigma_i} (1 - x_i) \leq w & \forall e \in E. \end{aligned}$$

2. $\widehat{x}_i$: value of x_i in the optimal LP solution.
3. $\widehat{w}$: value of w in LP solution.
4. Optimal congestion must be bigger than $\widehat{w}$.
5. X_i: random variable one with probability $\widehat{x}_i$, and zero otherwise.
6. If $X_i = 1$ then use π to route from $\mathbf{v}_i$ to $\mathbf{u}_i$.
7. Otherwise use σ_i.

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6. If $X_i = 1$ then use π to route from $\mathbf{v}_i$ to $\mathbf{u}_i$.
7. Otherwise use σ_i.

Minimizing congestion

1. IP $\implies$ LP:

$$\begin{aligned}
\textbf{min} \quad & w \\
\text{s.t.} \quad & x_i \geq 0 & i = 1, \ldots, t, \\
& x_i \leq 1 & i = 1, \ldots, t, \\
& \sum_{e \in \pi_i} x_i + \sum_{e \in \sigma_i} (1 - x_i) \leq w & \forall e \in E.
\end{aligned}$$

2. $\widehat{x_i}$: value of x_i in the optimal LP solution.
3. $\widehat{w}$: value of w in LP solution.
4. Optimal congestion must be bigger than $\widehat{w}$.
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1. Congestion of e is $Y_e = \sum_{e \in \pi_i} X_i + \sum_{e \in \sigma_i} (1 - X_i)$.
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$$\alpha_e = \mathbf{E}\left[Y_e\right] = \mathbf{E}\left[\sum_{e \in \pi_i} X_i + \sum_{e \in \sigma_i} (1 - X_i)\right]$$
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$$= \sum_{e \in \pi_i} \widehat{x}_i + \sum_{e \in \sigma_i} (1 - \widehat{x}_i) \leq \widehat{w}.$$

3. $\widehat{w}$: Fractional congestion (from LP solution).

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$$\Pr\Big[Y_e \geq (1+\delta)\alpha_e\Big] \leq \exp\left(-\frac{\alpha_e \delta^2}{4}\right) \leq \exp\left(-\frac{\widehat{w}\delta^2}{4}\right).$$

2. Let $\delta = \sqrt{\frac{400}{\widehat{w}} \ln t}$. We have that

$$\Pr\Big[Y_e \geq (1+\delta)\alpha_e\Big] \leq \exp\left(-\frac{\delta^2 \widehat{w}}{4}\right) \leq \frac{1}{t^{100}},$$

3. If $t \geq n^{1/50} \implies \forall$ edges in graph congestion $\leq (1+\delta)\widehat{w}$.
4. t: Number of pairs, n: Number of vertices in G.

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$$1 + \delta = 1 + \sqrt{\frac{20}{\widehat{w}} \ln t} \leq 1 + \frac{\sqrt{20 \ln n}}{n^{1/4}},$$

which is of course extremely close to 1, if n is sufficiently large.

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Theorem

1. G: *Graph* n *vertices.*
2. $(s_1, t_1), \ldots, (s_t, t_t)$: *pairs o vertices*
3. π_i, σ_i: *two different paths connecting* s_i *to* t_i
4. $\widehat{w}$: *Fractional congestion at least* $n^{1/2}$.
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6. $\implies$ *In polynomial time* (LP *solving time) choose paths*
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When the congestion is low

1. Assume $\widehat{w}$ is a constant.
2. Can get a better bound by using the Chernoff inequality in its more general form.
3. set $\delta = c \ln t / \ln \ln t$, where c is a constant. For $\mu = \alpha_e$, we have that

$$\Pr\left[Y_e \geq (1+\delta)\mu\right] \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu$$
$$= \exp\Bigl(\mu(\delta - (1+\delta)\ln(1+\delta))\Bigr)$$
$$= \exp\Bigl(-\mu c' \ln t\Bigr) \leq \frac{1}{t^{O(1)}},$$

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26.1.4: Reminder about Chernoff inequality

26.1.4.1: The Chernoff Bound — General Case

Chernoff inequality

Problem

Let $X_1, \ldots X_n$ be n independent Bernoulli trials, where

$$\Pr\left[X_i = 1\right] = p_i, \qquad \Pr\left[X_i = 0\right] = 1 - p_i,$$

$$Y = \sum_i X_i, \qquad \textit{and} \qquad \mu = \mathbf{E}\left[Y\right].$$

We are interested in bounding the probability that $Y \geq (1 + \delta)\mu$.

Chernoff inequality

Theorem (Chernoff inequality)

For any $\delta > 0$,

$$\Pr\left[Y > (1+\delta)\mu\right] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}.$$

Or in a more simplified form, for any $\delta \leq 2e - 1$,

$$\Pr\left[Y > (1+\delta)\mu\right] < \exp\left(-\mu\delta^2/4\right),$$

and

$$\Pr\left[Y > (1+\delta)\mu\right] < 2^{-\mu(1+\delta)},$$

for $\delta \geq 2e - 1$.

More Chernoff...

Theorem

Under the same assumptions as the theorem above, we have

$$\mathbf{Pr}\Big[Y < (1-\delta)\mu\Big] \leq \exp\left(-\mu\frac{\delta^2}{2}\right).$$

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