NP Completeness and Cook-Levin Theorem

Lecture 23
April 21, 2015

Turing Machines: Recap

- Infinite tape.
- Finite state control.
- Input at beginning of tape.
- Special tape letter “blank”.
- Head can move only one cell to left or right.

Turing Machines: Formally

- A TM $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$:
  - $Q$ is set of states in finite control
  - $q_0$ start state, $q_{accept}$ is accept state, $q_{reject}$ is reject state
  - $\Sigma$ is input alphabet, $\Gamma$ is tape alphabet (includes $\sqcup$)
  - $\delta : Q \times \Gamma \rightarrow \{L, R\} \times \Gamma \times Q$ is transition function
  - $\delta(q, a) = (q', b, L)$ means that $M$ in state $q$ and head seeing $a$ on tape will move to state $q'$ while replacing $a$ on tape with $b$ and head moves left.

- $L(M)$: language accepted by $M$ is set of all input strings $s$ on which $M$ accepts; that is:
  - TM is started in state $q_0$.
  - Initially, the tape head is located at the first cell.
  - The tape contain $s$ on the tape followed by blanks.
  - The TM halts in the state $q_{accept}$.

P and NP and Turing Machines

- Polynomial vs. polynomial time verifiable...
  - $P$: set of decision problems that have polynomial time algorithms.
  - $NP$: set of decision problems that have polynomial time non-deterministic algorithms.

- Question: What is an algorithm? Depends on the model of computation!
- What is our model of computation?
- Formally speaking our model of computation is Turing Machines.
P via TMs

- Polynomial time Turing machine.

Definition

$M$ is a polynomial time TM if there is some polynomial $p(\cdot)$ such that on all inputs $w$, $M$ halts in $p(|w|)$ steps.

- Polynomial time language.

Definition

$L$ is a language in P iff there is a polynomial time TM $M$ such that $L = L(M)$.

NP via TMs

- NP language...

Definition

$L$ is an NP language iff there is a non-deterministic polynomial time TM $M$ such that $L = L(M)$.

- Non-deterministic TM: each step has a choice of moves
  - $\delta : Q \times \Gamma \to 2^{(Q \times \Gamma \times \{L, R\})}$.
  - Example: $\delta(q, a) = \{(q_1, b, L), (q_2, c, R), (q_3, a, R)\}$ means that $M$ can non-deterministically choose one of the three possible moves from $(q, a)$.
  - $L(M)$: set of all strings $s$ on which there exists some sequence of valid choices at each step that lead from $q_0$ to $q_{accept}$.

Non-deterministic TMs vs certifiers

- Two definition of NP:
  - $L$ is in NP iff $L$ has a polynomial time certifier $C(\cdot, \cdot)$.
  - $L$ is in NP iff $L$ is decided by a non-deterministic polynomial time TM $M$.

- Equivalence...

Claim

Two definitions are equivalent.

- Why?
  - Informal proof idea: the certificate $t$ for $C$ corresponds to non-deterministic choices of $M$ and vice-versa.
  - In other words $L$ is in NP iff $L$ is accepted by a NTM which first guesses a proof $t$ of length poly in input $|s|$ and then acts as a deterministic TM.

Non-determinism, guessing and verification

- A non-deterministic machine has choices at each step and accepts a string if there exists a set of choices which lead to a final state.
- Equivalently the choices can be thought of as guessing a solution and then verifying that solution. In this view all the choices are made a priori and hence the verification can be deterministic. The "guess" is the "proof" and the "verifier" is the "certifier".
- Note: Symmetry inherent in the definition of non-determinism. Strings in the language can be easily verified. No easy way to verify that a string is not in the language.
Algorithms: **TM**s vs **RAM** Model

1. Why do we use **TM**s some times and **RAM** Model other times?
   - **TM**s are very simple: no complicated instruction set, no jumps/pointers, no explicit loops etc.
     - Simplicity is useful in proofs.
     - The “right” formal bare-bones model when dealing with subtleties.
2. **RAM** model is a closer approximation to the running time/space usage of realistic computers for reasonable problem sizes
   - Not appropriate for certain kinds of formal proofs when algorithms can take super-polynomial time and space

“Hardest” Problems

**Question**

- What is the hardest problem in **NP**? How do we define it?
- Towards a definition
  - Hardest problem must be in **NP**.
  - Hardest problem must be at least as “difficult” as every other problem in **NP**.

NP-Complete Problems

**Definition**

A problem $X$ is said to be **NP-Complete** if

1. $X \in \text{NP}$, and
2. (Hardness) For any $Y \in \text{NP}$, $Y \leq_p X$.

Solving NP-Complete Problems

**Proposition**

Suppose $X$ is **NP-Complete**. Then $X$ can be solved in polynomial time if and only if $P = \text{NP}$.

**Proof.**

$\Rightarrow$ Suppose $X$ can be solved in polynomial time

1. Let $Y \in \text{NP}$. We know $Y \leq_p X$.
2. We showed that if $Y \leq_p X$ and $X$ can be solved in polynomial time, then $Y$ can be solved in polynomial time.
3. Thus, every problem $Y \in \text{NP}$ is such that $Y \in P$; $\text{NP} \subseteq P$.
4. Since $P \subseteq \text{NP}$, we have $P = \text{NP}$.

$\Leftarrow$ Since $P = \text{NP}$, and $X \in \text{NP}$, we have a polynomial time algorithm for $X$. 

Output
NP-Hard Problems

- **NP-Hard** problems:

**Definition**

A problem $X$ is said to be **NP-Hard** if

1. (Hardness) For any $Y \in \text{NP}$, we have that $Y \leq_P X$.

- An **NP-Hard** problem need not be in **NP**!
- Example: Halting problem is **NP-Hard** (why?) but not **NP-Complete**.

Consequences of proving **NP-Completeness**

- If $X$ is **NP-Complete**
  1. Since we believe $P \neq \text{NP}$,
  2. and solving $X$ implies $P = \text{NP}$.
  3. $\implies X$ is unlikely to be efficiently solvable.
  4. $\implies$ At the very least, many smart people before you have failed to find an efficient algorithm for $X$.
  5. (This is proof by mob opinion — take with a grain of salt.)

NP-Complete Problems

**Question**

Are there any problems that are **NP-Complete**?

**Answer**

Yes! Many, many problems are **NP-Complete**.

Circuits

**Definition**

A circuit is a directed **acyclic** graph with

1. Input vertices (without incoming edges) labelled with 0, 1 or a distinct variable.
2. Every other vertex is labelled $\lor$, $\land$ or $\neg$.
3. Single node output vertex with no outgoing edges.
Cook-Levin Theorem

Definition (Circuit Satisfaction (CSAT).)
Given a circuit as input, is there an assignment to the input variables that causes the output to get value 1?

Theorem (Cook-Levin)
CSAT is NP-Complete.

Need to show
1. CSAT is in NP.
2. every NP problem X reduces to CSAT.

CSAT: Circuit Satisfaction

Claim
CSAT is in NP.

2. Certifier: Evaluate the value of each gate in a topological sort of DAG and check the output gate value.

CSAT is NP-hard: Idea

1. Need to show that every NP problem X reduces to CSAT.
2. What does it mean that X ∈ NP?
3. X ∈ NP implies that there are polynomials p() and q() and certifier/verifier program C such that for every string s the following is true:
   - If s is a YES instance (s ∈ X) then there is a proof t of length p(|s|) such that C(s, t) says YES.
   - If s is a NO instance (s ∉ X) then for every string t of length at p(|s|), C(s, t) says NO.
   - C(s, t) runs in time q(|s| + |t|) time (hence polynomial time).

Reducing X to CSAT

1. X is in NP means we have access to p(), q(), C(·, ·).
2. What is C(·, ·)? It is a program or equivalently a Turing Machine!
3. How are p() and q() given? As numbers (coefficients and powers).
4. Example: if 3 is given then p(n) = n^3.
5. Thus an NP problem is essentially a three tuple <p, q, C> where C is either a program or a TM.
Reducing $X$ to CSAT

- Thus an NP problem is essentially a three tuple $\langle p, q, C \rangle$ where $C$ is either a program or TM.
- Problem $X$: Given string $s$, is $s \in X$?
- Same as the following: is there a proof $t$ of length $p(|s|)$ such that $C(s, t)$ says YES.
- How do we reduce $X$ to CSAT? Need an algorithm $A$ that
  - takes $s$ (and $\langle p, q, C \rangle$) and creates a circuit $G$ in polynomial time in $|s|$ (note that $\langle p, q, C \rangle$ are fixed).
  - $G$ is satisfiable if and only if there is a proof $t$ such that $C(s, t)$ says YES.

Simple but Big Idea: Programs are essentially the same as Circuits!

- Convert $C(s, t)$ into a circuit $G$ with $t$ as unknown inputs (rest is known including $s$)
- We know that $|t| = p(|s|)$ so express boolean string $t$ as $p(|s|)$ variables $t_1, t_2, \ldots, t_k$ where $k = p(|s|)$.
- Asking if there is a proof $t$ that makes $C(s, t)$ say YES is same as whether there is an assignment of values to “unknown” variables $t_1, t_2, \ldots, t_k$ that will make $G$ evaluate to true/YES.

Example: Independent Set

- Problem: Does $G = (V, E)$ have an Independent Set of size $\geq k$?
  - Certificate: Set $S \subseteq V$.
  - Certifier: Check $|S| \geq k$ and no pair of vertices in $S$ is connected by an edge.
- Formally, why is Independent Set in NP?

Formally why is Independent Set in NP?

- Input: $\langle n, y_{1,1}, y_{1,2}, \ldots, y_{1,n}, y_{2,1}, \ldots, y_{2,n}, \ldots, y_{n,1}, \ldots, y_{n,n}, k \rangle$ encodes $\langle G, k \rangle$.
  - $n$ is number of vertices in $G$
  - $y_{i,j}$ is a bit which is 1 if edge $(i, j)$ is in $G$ and 0 otherwise (adjacency matrix representation)
  - $k$ is size of independent set.
- Certificate: $t = t_1 t_2 \ldots t_n$. Interpretation is that $t_i$ is 1 if vertex $i$ is in the independent set, 0 otherwise.
Certifier for Independent Set

Certifier $C(s, t)$ for Independent Set:

if $(t_1 + t_2 + \ldots + t_n < k)$ then
  return NO
else
  for each $(i, j)$ do
    if $(t_i \land t_j \land \gamma_{i,j})$ then
      return NO
  return YES

Example: Independent Set

A certifier circuit for Independent Set

Programs, Turing Machines and Circuits

1. Consider “program” $A$ that takes $f(|s|)$ steps on input string $s$.
2. Question: What computer is the program running on and what does step mean?
3. Real computers difficult to reason with mathematically because
   a. instruction set is too rich
   b. pointers and control flow jumps in one step
   c. assumption that pointer to code fits in one word
4. Turing Machines
   a. simpler model of computation to reason with
   b. can simulate real computers with polynomial slow down
   c. all moves are local (head moves only one cell)

Certifiers that at TMs

1. Assume $C(\cdot, \cdot)$ is a (deterministic) Turing Machine $M$
2. Problem: Given $M$, input $s$, $p$, $q$ decide if there is a proof $t$ of length $p(|s|)$ such that $M$ on $s$, $t$ will halt in $q(|s|)$ time and say YES.
3. There is an algorithm $A$ that can reduce above problem to CSAT mechanically as follows.
   a. $A$ first computes $p(|s|)$ and $q(|s|)$.
   b. Knows that $M$ can use at most $q(|s|)$ memory/tape cells
   c. Knows that $M$ can run for at most $q(|s|)$ time
   d. Simulates the evolution of the state of $M$ and memory over time using a big circuit.
Simulation of Computation via Circuit

1. Think of $M$’s state at time $\ell$ as a string $x^\ell = x_1 x_2 \ldots x_k$ where each $x_i \in \{0, 1, B\} \times Q \cup \{q_{-1}\}$. 
2. At time $0$ the state of $M$ consists of input string $s$ a guess $t$ (unknown variables) of length $p(|s|)$ and rest $q(|s|)$ blank symbols. 
3. At time $q(|s|)$ we wish to know if $M$ stops in $q_{\text{accept}}$ with say all blanks on the tape. 
4. We write a circuit $C_\ell$ which captures the transition of $M$ from time $\ell$ to time $\ell + 1$. 
5. Composition of the circuits for all times $0$ to $q(|s|)$ gives a big (still poly) sized circuit $C$. 
6. The final output of $C$ should be true if and only if the entire state of $M$ at the end leads to an accept state.

NP-Hardness of Circuit Satisfaction

1. Key Ideas in reduction: 
   - Use TMs as the code for certifier for simplicity 
   - Since $p()$ and $q()$ are known to $A$, it can set up all required memory and time steps in advance 
   - Simulate computation of the TM from one time to the next as a circuit that only looks at three adjacent cells at a time 
   - Note: Above reduction can be done to SAT as well. Reduction to SAT was the original proof of Steve Cook.

SAT is NP-Complete

1. We have seen that SAT $\in$ NP 
2. To show NP-Hardness, we will reduce Circuit Satisfiability (CSAT) to SAT 
   - Instance of CSAT (we label each node):

Converting a circuit into a CNF formula

Label the nodes

(A) Input circuit
(B) Label the nodes.
Converting a circuit into a CNF formula

Introduce a variable for each node

(B) Label the nodes.  
(C) Introduce var for each node.

Converting a circuit into a CNF formula

Convert each sub-formula to an equivalent CNF formula

<table>
<thead>
<tr>
<th>$x_k$</th>
<th>$x_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_k = x_i \land x_j$</td>
<td>$(\neg x_k \lor x_j) \land (\neg x_k \lor x_j) \land (x_k \lor \neg x_j \lor \neg x_j)$</td>
</tr>
<tr>
<td>$x_j = x_g \land x_h$</td>
<td>$(\neg x_j \lor x_g) \land (\neg x_j \lor x_g) \land (x_j \lor \neg x_g \lor \neg x_g)$</td>
</tr>
<tr>
<td>$x_i = \neg x_f$</td>
<td>$(x_i \lor \neg x_f) \land (\neg x_i \lor \neg x_f)$</td>
</tr>
<tr>
<td>$x_h = x_d \land x_e$</td>
<td>$(x_h \lor \neg x_d) \land (x_h \lor \neg x_e) \land (\neg x_h \lor x_d \lor x_e)$</td>
</tr>
<tr>
<td>$x_g = x_b \land x_c$</td>
<td>$(x_g \lor \neg x_b) \land (x_g \lor \neg x_c) \land (\neg x_g \lor x_b \lor x_c)$</td>
</tr>
<tr>
<td>$x_f = x_a \land x_b$</td>
<td>$(\neg x_f \lor x_a) \land (\neg x_f \lor x_b) \land (x_f \lor \neg x_a \lor \neg x_b)$</td>
</tr>
<tr>
<td>$x_d = 0$</td>
<td>$\neg x_d$</td>
</tr>
<tr>
<td>$x_a = 1$</td>
<td>$x_a$</td>
</tr>
</tbody>
</table>

Converting a circuit into a CNF formula

Write a sub-formula for each variable that is true if the var is computed correctly.

$\neg x_k$ (Demand a sat' assignment!)

$\neg x_k = x_i \land x_k$
$\neg x_j = x_g \land x_h$
$\neg x_i = \neg x_f$
$\neg x_h = x_d \lor x_e$
$\neg x_g = x_b \lor x_c$
$\neg x_f = x_a \land x_b$
$\neg x_d = 0$
$x_a = 1$

(D) Write a sub-formula for each variable that is true if the var is computed correctly.

Converting a circuit into a CNF formula

Take the conjunction of all the CNF sub-formulas

$\neg x_k \land (\neg x_k \lor x_j) \land (\neg x_k \lor x_j) \land (x_k \lor \neg x_j \lor \neg x_j)$
$\land (\neg x_j \lor x_h) \land (\neg x_j \lor x_h) \land (x_j \lor \neg x_h \lor \neg x_h)$
$\land (\neg x_i \lor x_f) \land (\neg x_i \lor x_f)$
$\land (\neg x_h \lor x_d) \land (\neg x_h \lor x_d) \land (\neg x_h \lor x_d \lor x_e)$
$\land (\neg x_g \lor x_b) \land (\neg x_g \lor x_b) \land (\neg x_g \lor x_b \lor x_c)$
$\land (\neg x_f \lor x_a) \land (\neg x_f \lor x_b) \land (\neg x_f \lor \neg x_a \lor \neg x_b)$
$\land (\neg x_d \lor \neg x_a \lor \neg x_b) \land (\neg x_d \land x_a)$

We got a CNF formula that is satisfiable if and only if the original circuit is satisfiable.
Reduction: \( \text{CSAT} \leq_p \text{SAT} \)

For each gate (vertex) \( v \) in the circuit, create a variable \( x_v \)

**Case \( \neg \):** \( v \) is labeled \( \neg \) and has one incoming edge from \( u \) (so \( x_v = \neg x_u \)). In SAT formula generate, add clauses \((x_u \lor x_v)\), \((\neg x_u \lor \neg x_v)\). Observe that

\[
x_v = \neg x_u \text{ is true } \iff (x_u \lor x_v) \text{ both true.}
\]

Reduction: \( \text{CSAT} \leq_p \text{SAT} \)

Continued...

1. **Case \( \lor \):** So \( x_v = x_u \lor x_w \). In SAT formula generated, add clauses \((x_v \lor \neg x_u)\), \((x_v \lor \neg x_w)\), and \((\neg x_v \lor x_u \lor x_w)\). Again, observe that

\[
(x_v = x_u \lor x_w) \text{ is true } \iff (x_v \lor \neg x_u), (x_v \lor \neg x_w), \text{ all true.}
\]

Reduction: \( \text{CSAT} \leq_p \text{SAT} \)

Continued...

If \( v \) is an input gate with a fixed value then we do the following.

- If \( x_v = 1 \) add clause \( x_v \).
- If \( x_v = 0 \) add clause \( \neg x_v \).

Add the clause \( x_v \) where \( v \) is the variable for the output gate.
Correctness of Reduction

Need to show circuit $C$ is satisfiable iff $\varphi_C$ is satisfiable

$\Rightarrow$ Consider a satisfying assignment $a$ for $C$
  1. Find values of all gates in $C$ under $a$
  2. Give value of gate $v$ to variable $x_v$; call this assignment $a'$
  3. $a'$ satisfies $\varphi_C$ (exercise)

$\Leftarrow$ Consider a satisfying assignment $a$ for $\varphi_C$
  1. Let $a'$ be the restriction of $a$ to only the input variables
  2. Value of gate $v$ under $a'$ is the same as value of $x_v$ in $a$
  3. Thus, $a'$ satisfies $C$

Proving that a problem $X$ is NP-Complete

1. To prove $X$ is NP-Complete, show
   1. Show $X$ is in NP.
      1. certificate/proof of polynomial size in input
      2. polynomial time certifier $C(s, t)$
      3. Reduction from a known NP-Complete problem such as CSAT or SAT to $X$
   2. $\text{SAT} \leq_p X$ implies that every NP problem $Y \leq_p X$. Why?
   3. Transitivity of reductions:
      1. $Y \leq_p \text{SAT}$ and $\text{SAT} \leq_p X$ and hence $Y \leq_p X$.

NP-Completeness via Reductions

1. What we know so far:
   1. CSAT is NP-Complete.
   2. CSAT $\leq_p \text{SAT}$ and SAT is in NP and hence SAT is NP-Complete.
   3. SAT $\leq_p 3\text{-SAT}$ and hence 3-SAT is NP-Complete.
   4. 3-SAT $\leq_p \text{Independent Set}$ (which is in NP) and hence Independent Set is NP-Complete.
   5. Vertex Cover is NP-Complete.
   6. Clique is NP-Complete.
   7. Gazillion of different problems from many areas of science and engineering have been shown to be NP-Complete.
   8. A surprisingly frequent phenomenon!