Chapter 19

Applications of Network Flows

OLD CS 473: Fundamental Algorithms, Spring 2015
April 2, 2015

19.0.1 Important Properties of Flows

19.0.1.1 Network flow, what we know...

(A) G: Network flow with n vertices and m edges.
(B) **algFordFulkerson** computes max-flow if capacities are integers.
(C) If total capacity is C, running time of **algFordFulkerson** is $O(mC)$.
(D) **algFordFulkerson** is not polynomial time.
(E) **algFordFulkerson** might not terminate if capacities are real numbers.
(F) ...see end of the slides in previous lectures for detailed example.

19.1 Edmonds-Karp algorithm

19.1.0.2 Edmonds-Karp algorithm

```
algEdmondsKarp
    for every edge e, f(e) = 0
    G_f is residual graph of G with respect to f
    while G_f has a simple s-t path do
        Perform BFS in G_f
        P: shortest s-t path in G_f
        f = augment(f, P)
        Construct new residual graph G_f.
```

Theorem 19.1.1. *Given a network flow G with n vertices and m edges, and capacities that are real numbers, the algorithm* **algEdmondsKarp** *computes the maximum flow in G.*

The running time is $O(m^2 n)$.

19.1.1 Computing a minimum cut...

19.1.1.1 Finding a Minimum Cut

(A) **Question:** How do we find an actual minimum s-t cut?

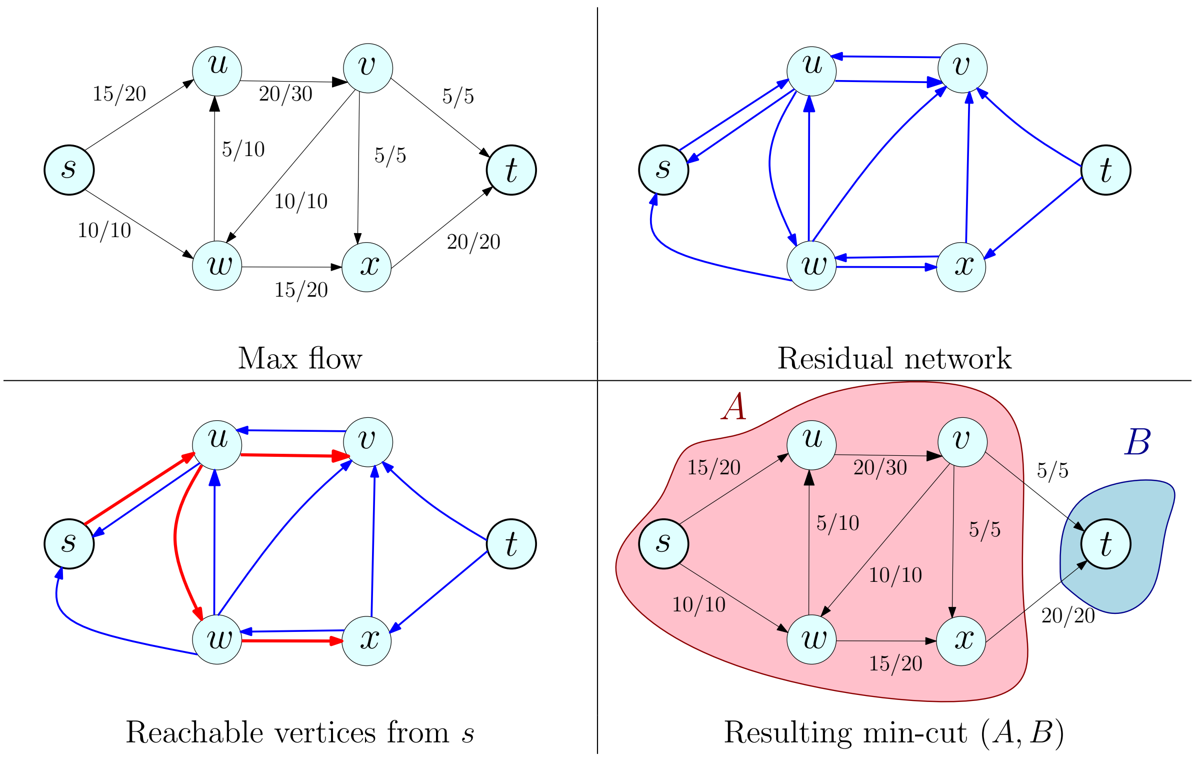
(B) Proof gives the algorithm!

- (A) Compute an s-t maximum flow f in G
- (B) Obtain the residual graph G_f
- (C) Find the nodes A reachable from s in G_f
- (D) Output the cut $(A, B) = \{(u, v) \mid u \in A, v \in B\}$. **Note:** The cut is found in G while A is found in G_f

(C) Running time is essentially the same as finding a maximum flow.

(D) **Note:** Given G and a flow f there is a linear time algorithm to check if f is a maximum flow and if it is, outputs a minimum cut. How?

19.1.1.2 Min cut from max-flow



Max flow — Residual network — Reachable vertices from s — Resulting min-cut (A, B)

19.1.1.3 Network Flow: Facts to Remember

Flow network: directed graph G, capacities c, source s, sink t.

(A) Maximum s-t flow can be computed:

- (A) Using Ford-Fulkerson algorithm in $O(mC)$ time when capacities are integral and C is an upper bound on the flow.
- (B) Using variant of algorithm, in $O(m^2 \log C)$ time, when capacities are integral. (Polynomial time.)

(C) Using Edmonds-Karp algorithm, in $O(m^2 n)$ time, when capacities are rational (strongly polynomial time algorithm).

19.1.2 Network Flow

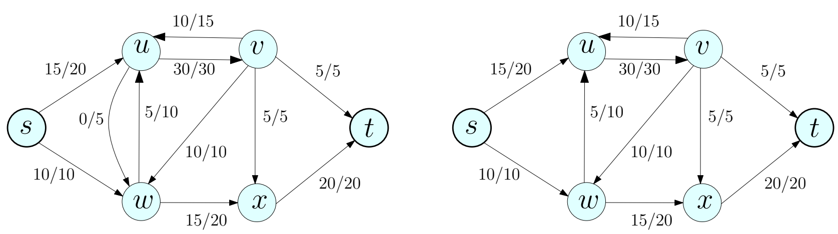
19.1.2.1 Even more facts to remember

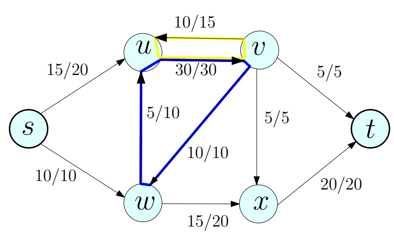
(A) If capacities are integral then there is a maximum flow that is integral and above algorithms give an integral max flow. This is known as ***integrality of flow***.
(B) Given a flow of value v, can decompose into $O(m+n)$ flow paths of same total value v. Integral flow implies integral flow on paths.
(C) Maximum flow is equal to the minimum cut and minimum cut can be found in $O(m+n)$ time given any maximum flow.

19.1.2.2 Paths, Cycles and Acyclicity of Flows

Definition 19.1.2. *Given a flow network $G = (V, E)$ and a flow $f : E \to \mathbb{R}^{\geq 0}$ on the edges, the* **support** *of f is the set of edges $E' \subseteq E$ with non-zero flow on them. That is, $E' = \{e \in E \mid f(e) > 0\}$.*

Question: Given a flow f, can there by cycles in its support?





19.1.2.3 Acyclicity of Flows

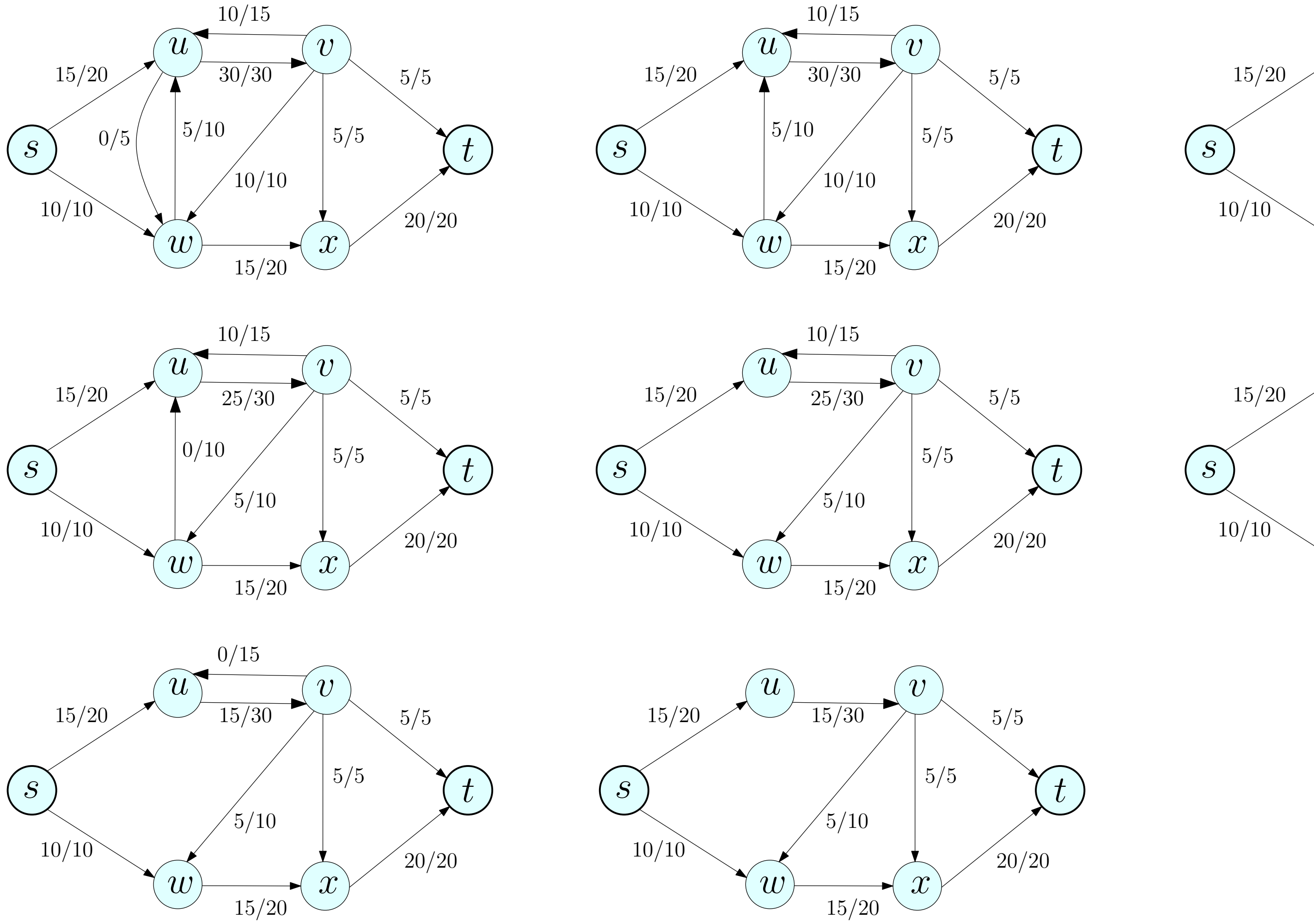
Proposition 19.1.3. *In any flow network, if f is a flow then there is another flow f' such that the support of f' is an* acyclic *graph and $v(f') = v(f)$. Further if f is an integral flow then so is f'.*

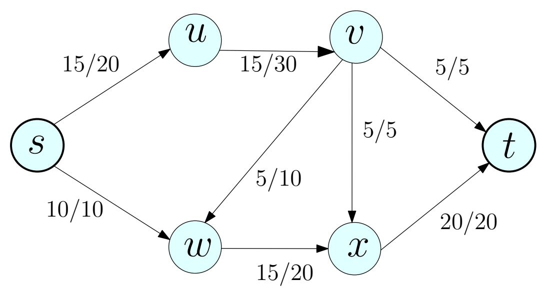
Proof:
(A) $E' = \{e \in E \mid f(e) > 0\}$, support of f.
(B) Suppose there is a directed cycle C in E'

(C) Let e' be the edge in C with least amount of flow
(D) For each $e \in C$, reduce flow by $f(e')$. Remains a flow. Why?
(E) Flow on e' is reduced to 0.
(F) Claim: Flow value from s to t does not change. Why?
(G) Iterate until no cycles

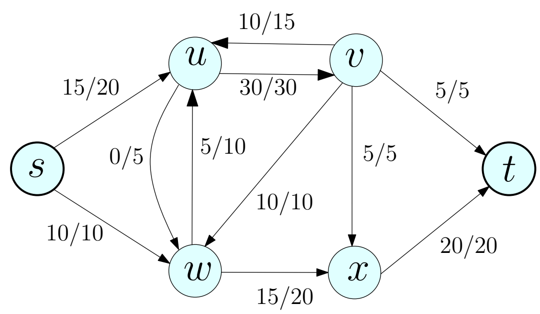
∎

19.1.2.4 Example





Throw away edge with no flow on itFind a cycle in the support/flowReduce flow on cycle as much as possibleThrow away edge with no flow on itFind a cycle in the support/flowReduce flow on cycle as much as possibleThrow away edge with no flow on itViola!!! An equivalent flow with no cycles in it. Original flow:



19.1.2.5 Flow Decomposition

Lemma 19.1.4. *Given an edge based flow $f : E \to \mathbb{R}^{\geq 0}$, there exists a collection of paths $\mathcal{P}$ and cycles $\mathcal{C}$ and an assignment of flow to them $f' : \mathcal{P} \cup \mathcal{C} \to \mathbb{R}^{\geq 0}$ such that:*

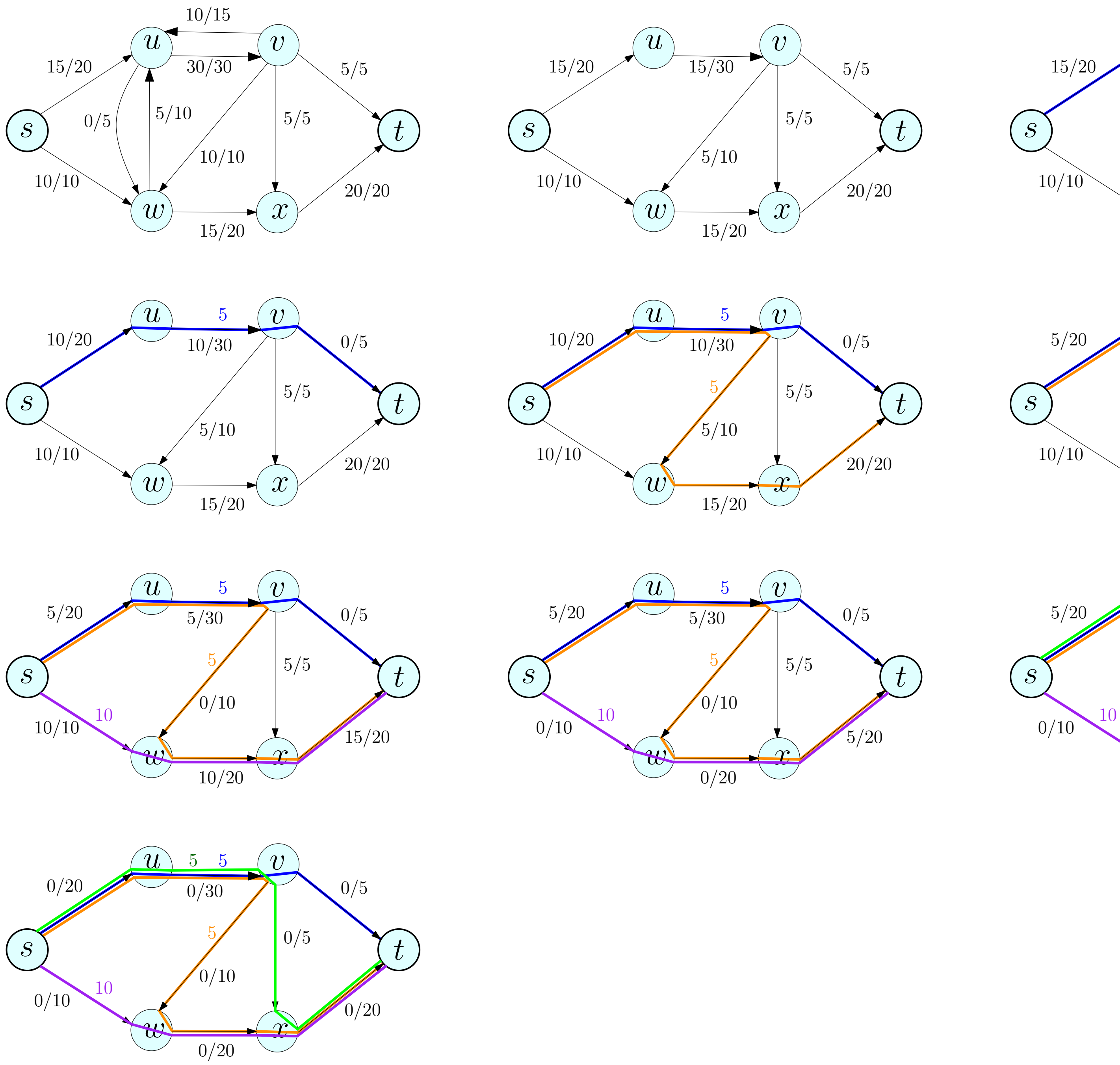
(A) $|\mathcal{P} \cup \mathcal{C}| \leq m$
(B) for each $e \in E$, $\sum_{P \in \mathcal{P}: e \in P} f'(P) + \sum_{C \in \mathcal{C}: e \in C} f'(C) = f(e)$
(C) $v(f) = \sum_{P \in \mathcal{P}} f'(P)$.
(D) if f is integral then so are $f'(P)$ and $f'(C)$ for all P and C

Proof:[Proof Idea]

(A) Remove all cycles as in previous proposition.
(B) Next, decompose into paths as in previous lecture.
(C) Exercise: verify claims.

■

19.1.2.6 Example



Find cycles as shown beforeFind a source to sink path, and push max flow along it (5 unites)Compute remaining flowFind a source to sink path, and push max flow along it (5 unites). Edges with 0 flow on them can not be used as they are no longer in the support of the flow.Compute remaining flowFind a source to sink path, and push max flow along it (10 unites). Compute remaining flowFind a source to sink path, and push max flow along it (5 unites). Compute remaining flowNo flow remains in the graph. We fully decomposed the flow into flow on paths. Together with the cycles, we get a decomposition of the original flow into m flows on paths and cycles.

19.1.2.7 Flow Decomposition

Lemma 19.1.5. *Given an edge based flow $f : E \to \mathbb{R}^{\geq 0}$, there exists a collection of paths $\mathcal{P}$ and cycles $\mathcal{C}$ and an assignment of flow to them $f' : \mathcal{P} \cup \mathcal{C} \to \mathbb{R}^{\geq 0}$ such that:*
(A) $|\mathcal{P} \cup \mathcal{C}| \leq m$
(B) for each $e \in E$, $\sum_{P \in \mathcal{P}: e \in P} f'(P) + \sum_{C \in \mathcal{C}: e \in C} f'(C) = f(e)$
(C) $v(f) = \sum_{P \in \mathcal{P}} f'(P)$.
(D) if f is integral then so are $f'(P)$ and $f'(C)$ for all P and C.
Above flow decomposition can be computed in $O(m^2)$ time.

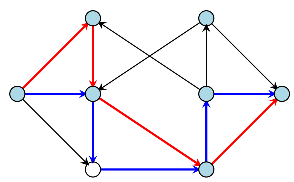
19.2 Network Flow Applications I

19.2.1 Edge Disjoint Paths

19.2.2 Directed Graphs

19.2.2.1 Edge-Disjoint Paths in Directed Graphs

Definition 19.2.1.



A set of paths is **edge disjoint** *if no two path*
an edge.

Problem Given a directed graph with two special vertices s and t, find the *maximum* number of edge disjoint paths from s to t. **Applications:** Fault tolerance in routing — edges/nodes in networks can fail. Disjoint paths allow for planning backup routes in case of failures.

19.2.3 Reduction to Max-Flow

19.2.3.1 Reduction to Max-Flow

Problem Given a directed graph G with two special vertices s and t, find the maximum number of edge disjoint paths from s to t. Reduction Consider G as a flow network with edge capacities 1, and compute max-flow.

19.2.3.2 Correctness of Reduction

Lemma 19.2.2. *If G has k edge disjoint paths $P_1, P_2, \ldots, P_k$ then there is an s-t flow of value k in G.*

Proof: Set $f(e) = 1$ if e belongs to one of the paths $P_1, P_2, \ldots, P_k$; other-wise set $f(e) = 0$. This defines a flow of value k. ∎

19.2.3.3 Correctness of Reduction

Lemma 19.2.3. *If G has a flow of value k then there are k edge disjoint paths between s and t.*

Proof:
(A) Capacities are all 1 and hence there is integer flow of value k, that is $f(e) = 0$ or $f(e) = 1$ for each e.
(B) Decompose flow into paths.
(C) Flow on each path is either 1 or 0.
(D) Hence there are k paths $P_1, P_2, \ldots, P_k$ with flow of 1 each.
(E) Paths are edge-disjoint since capacities are 1.

■

19.2.3.4 Running Time

Theorem 19.2.4. *The number of edge disjoint paths in G can be found in $O(mn)$ time.*

Proof:
(A) Set capacities of edges in G to 1.
(B) Run Ford-Fulkerson algorithm.
(C) Maximum value of flow is n and hence run-time is $O(nm)$.
(D) Decompose flow into k paths ($k \leq n$).
Takes $O(k \times m) = O(km) = O(mn)$ time.

■

Remark Algorithm computes set of edge-disjoint paths realizing opt. solution.

19.2.4 Menger's Theorem

19.2.4.1 Menger's Theorem

Theorem 19.2.5 (Menger [1927]). *Let G be a directed graph. The minimum number of edges whose removal disconnects s from t (the minimum-cut between s and t) is equal to the maximum number of edge-disjoint paths in G between s and t.*

Proof: Maxflow-mincut theorem and integrality of flow. ■

Menger proved his theorem before Maxflow-Mincut theorem! Maxflow-Mincut theorem is a generalization of Menger's theorem to capacitated graphs.

19.2.5 Undirected Graphs

19.2.5.1 Edge Disjoint Paths in Undirected Graphs

(A) The problem: Problem Given an **undirected** graph G, find the maximum number of edge disjoint paths in G

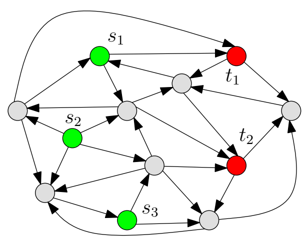
(B) Reduction:
   (A) create **directed** graph H by adding directed edges (u, v) and (v, u) for each edge uv in G.
   (B) compute maximum s-t flow in H.

(C) **Problem:** Both edges (u, v) and (v, u) may have non-zero flow!

(D) **Not a Problem!** Can assume maximum flow in H is acyclic and hence cannot have non-zero flow on both (u, v) and (v, u). Reduction works. See book for more details.

19.2.6 Multiple Sources and Sinks

19.2.6.1 Multiple Sources and Sinks



(A) Input:
   (A) A directed graph G with edge capacities $c(e)$.
   (B) Source nodes $s_1, s_2, \ldots, s_k$.
   (C) Sink nodes $t_1, t_2, \ldots, t_\ell$.
   (D) Sources and sinks are *disjoint*.

(A) **Maximum Flow**: Send as much flow as possible from the sources to the sinks. *Sinks don't care which source they get flow from.*

(B) **Minimum Cut**: Find a minimum capacity set of edge E' such that removing E' disconnects every source from every sink.

19.2.6.2 Multiple Sources and Sinks: Formal Definition

(A) Input:
   (A) A directed graph G with edge capacities $c(e)$.
   (B) Source nodes $s_1, s_2, \ldots, s_k$.
   (C) Sink nodes $t_1, t_2, \ldots, t_\ell$.
   (D) Sources and sinks are *disjoint*.

(B) A function $f : E \to \mathbb{R}^{\geq 0}$ is a ***flow*** if:
   (A) For each $e \in E$, $f(e) \leq c(e)$, and
   (B) for each v which is not a source or a sink $f^{\text{in}}(v) = f^{\text{out}}(v)$.

(C) **Goal:** max $\sum_{i=1}^{k}(f^{\text{out}}(s_i) - f^{\text{in}}(s_i))$, that is, flow out of sources.

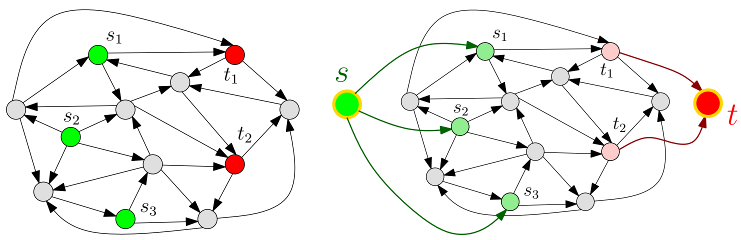
19.2.6.3 Reduction to Single-Source Single-Sink

(A) Add a *source* node s and a sink node t.

(B) Add edges $(s, s_1), (s, s_2), \ldots, (s, s_k)$.

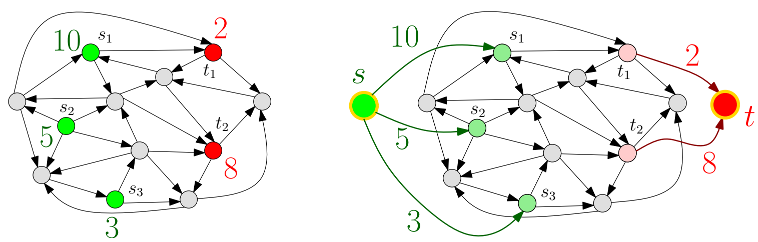
(C) Add edges $(t_1, t), (t_2, t), \ldots, (t_\ell, t)$.

(D) Set the capacity of the new edges to be ∞.



19.2.6.4 Supplies and Demands

(A) A further generalization:
  (A) source s_i has a supply of $S_i \geq 0$
  (B) since t_j has a demand of $D_j \geq 0$ units
(B) **Question:** is there a flow from source to sinks such that supplies are not exceeded and demands are met?
(C) Formally: additional constraints that $f^{\text{out}}(s_i) - f^{\text{in}}(s_i) \leq S_i$ for each source s_i and $f^{\text{in}}(t_j) - f^{\text{out}}(t_j) \geq D_j$ for each sink t_j.



19.2.7 Bipartite Matching

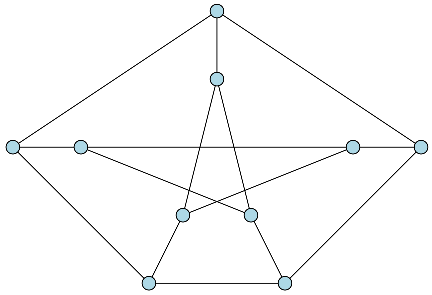
19.2.8 Definitions

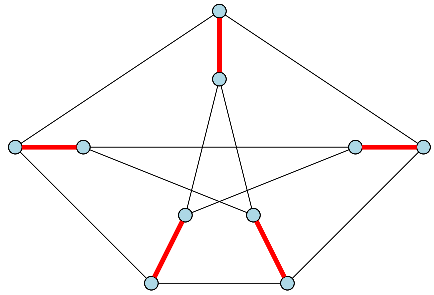
19.2.8.1 Matching

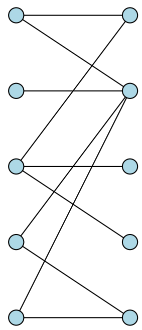
Problem 19.2.6 (Matching).
Input: *Given a (undirected) graph $G = (V, E)$.*
Goal: *Find a matching of maximum cardinality.*
  (A) A matching is $M \subseteq E$ such that at most one edge in M is incident on any vertex

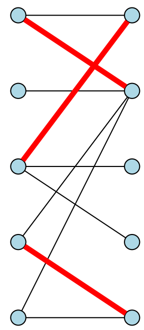
19.2.8.2 Bipartite Matching

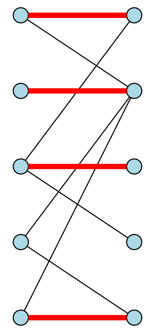
Problem 19.2.7 (Bipartite matching).
Input: *Given a bipartite graph $G = (L \cup R, E)$.*
Goal: *Find a matching of maximum cardinality*









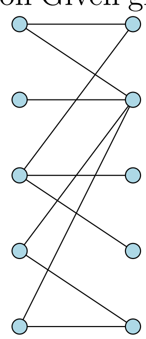


Maximum matching has 4 edges

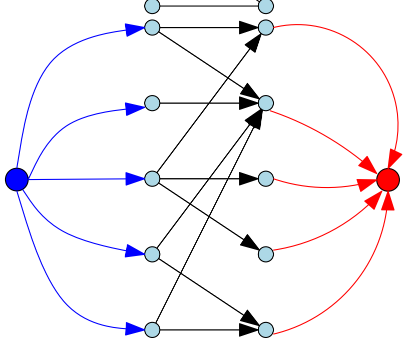
19.2.9 Reduction of bipartite matching to max-flow

19.2.9.1 Reduction of bipartite matching to max-flow

Max-Flow Construction Given graph $G = (L \cup R, E)$ create flow-network $G' = (V', E')$ as fol-



lows:



(A) $V' = L \cup R \cup \{s, t\}$ where s and t are the new source and sink.

(B) Direct all edges in E from L to R, and add edges from s to all vertices in L and from each vertex in R to t.

(C) Capacity of every edge is 1.

19.2.9.2 Correctness: Matching to Flow

Proposition 19.2.8. *If G has a matching of size k then G' has a flow of value k.*

Proof: Let M be matching of size k. Let $M = \{(u_1, v_1), \ldots, (u_k, v_k)\}$. Consider following flow f in G':
(A) $f(s, u_i) = 1$ and $f(v_i, t) = 1$ for $1 \leq i \leq k$
(B) $f(u_i, v_i) = 1$ for $1 \leq i \leq k$
(C) for all other edges flow is zero.
Verify that f is a flow of value k (because M is a matching). ■

19.2.9.3 Correctness: Flow to Matching

Proposition 19.2.9. *If G' has a flow of value k then G has a matching of size k.*

Proof: Consider flow f of value k.
(A) Can assume f is integral. Thus each edge has flow 1 or 0.
(B) Consider the set M of edges from L to R that have flow 1.
  (A) M has k edges because value of flow is equal to the number of non-zero flow edges crossing cut $(L \cup \{s\}, R \cup \{t\})$
  (B) Each vertex has at most one edge in M incident upon it. Why?

■

19.2.9.4 Correctness of Reduction

Theorem 19.2.10. *The maximum flow value in G' = maximum cardinality of matching in G.*

Consequence Thus, to find maximum cardinality matching in G, we construct G' and find the maximum flow in G'. Note that the matching itself (not just the value) can be found efficiently from the flow.

19.2.9.5 Running Time

For graph G with n vertices and m edges G' has $O(n + m)$ edges, and $O(n)$ vertices.
(A) Generic Ford-Fulkerson: Running time is $O(mC) = O(nm)$ since $C = n$.
(B) Capacity scaling: Running time is $O(m^2 \log C) = O(m^2 \log n)$.
Better running time is known: $O(m\sqrt{n})$.

19.2.10 Perfect Matchings

19.2.10.1 Perfect Matchings

Definition 19.2.11. *A matching M is* **perfect** *if every vertex has one edge in M incident upon it.*

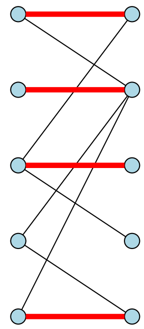


Figure 19.1: This graph does not have a perfect matching

19.2.10.2 Characterizing Perfect Matchings

Problem When does a bipartite graph have a perfect matching?
(A) Clearly $|L| = |R|$
(B) Are there any necessary and sufficient conditions?

19.2.10.3 A Necessary Condition

Lemma 19.2.12. *If $G = (L \cup R, E)$ has a perfect matching then for any $X \subseteq L$, $|N(X)| \geq |X|$, where $N(X)$ is the set of neighbors of vertices in X.*

Proof: Since G has a perfect matching, every vertex of X is matched to a different neighbor, and so $|N(X)| \geq |X|$. ∎

19.2.10.4 Hall's Theorem

(A) Frobenius-Hall theorem:

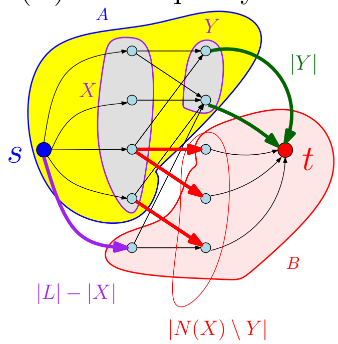
> **Theorem 19.2.13 ().** *Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. G has a perfect matching if and only if for every $X \subseteq L$, $|N(X)| \geq |X|$.*

(B) One direction is the necessary condition.
(C) For the other direction we will show the following:
  (A) Create flow network G' from G.
  (B) If $|N(X)| \geq |X|$ for all X, show that minimum s-t cut in G' is of capacity $n = |L| = |R|$.
  (C) Implies that G has a perfect matching.

19.2.10.5 Proof of Sufficiency

(A) Assume $|N(X)| \geq |X|$ for any $X \subseteq L$. Then show that min s-t cut in G' is of capacity at least n.
(B) Let (A, B) be an *arbitrary* s-t cut in G'
  (A) Let $X = A \cap L$ and $Y = A \cap R$.

(B) Cut capacity is at least $(|L| - |X|) + |Y| + |N(X) \setminus Y|$



Because there are...
(A) $|L| - |X|$ edges from s to $L \cap B$.
(B) $|Y|$ edges from Y to t.
(C) there are at least $|N(X) \setminus Y|$ edges from X to vertices on the right side that are not in Y.

19.2.11 Proof of Sufficiency

19.2.11.1 Continued...

(A) By the above, cut capacity is at least
$$\alpha = (|L| - |X|) + |Y| + |N(X) \setminus Y|.$$
(B) $|N(X) \setminus Y| \geq |N(X)| - |Y|$.
(This holds for any two sets.)
(C) By assumption $|N(X)| \geq |X|$ and hence
$$|N(X) \setminus Y| \geq |N(X)| - |Y| \geq |X| - |Y|.$$
(D) Cut capacity is therefore at least
$$\begin{aligned}\alpha &= (|L| - |X|) + |Y| + |N(X) \setminus Y| \\ &\geq |L| - |X| + |Y| + |X| - |Y| \geq |L| = n.\end{aligned}$$
(E) Any s-t cut capacity is at least $n \implies$ max flow at least n units $\implies$ perfect matching. **QED**

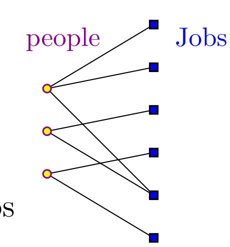
19.2.11.2 Hall's Theorem: Generalization

Theorem 19.2.14 (Frobenius-Hall). *Let $G = (L \cup R, E)$ be a bipartite graph with $|L| \leq |R|$. G has a matching that matches all nodes in L if and only if for every $X \subseteq L$, $|N(X)| \geq |X|$.*

Proof is essentially the same as the previous one.

19.2.11.3 Problem: Assigning jobs to people

Problem:
(A) n jobs or tasks
(B) m people.
(C) for each job a set of people who can do that job.
(D) for each person j a limit on number of jobs k_j.
(E) **Goal:** find an assignment of jobs to people so that all jobs are assigned and no person is overloaded.



19.2.11.4 Application: Assigning jobs to people

(A) Reduce to max-flow similar to matching.
(B) Arises in many settings. Using *minimum-cost flows* can also handle the case when assigning a job i to person j costs c_{ij} and goal is assign all jobs but minimize cost of assignment.

19.2.12 Reduction to Maximum Flow

19.2.12.1 For assigning jobs to people

(A) Create directed graph $G = (V, E)$ as follows
  (A) $V = \{s, t\} \cup L \cup R$: L set of n jobs, R set of m people
  (B) add edges (s, i) for each job $i \in L$, capacity 1
  (C) add edges (j, t) for each person $j \in R$, capacity k_j
  (D) if job i can be done by person j add an edge (i, j), capacity 1
(B) Compute max s-t flow. There is an assignment if and only if flow value is n.

19.2.12.2 Matchings in General Graphs

(A) Matchings in general graphs more complicated.
(B) There is a polynomial time algorithm to compute a maximum matching in a general graph. Best known running time is $O(m\sqrt{n})$.

Bibliography

K. Menger. Zur allgemeinen kruventheorie. *Fund. Math.*, 10:96–115, 1927.