Hashing
Lecture 16
March 17, 2015
Dictionary Data Structure

1. $\mathcal{U}$: universe of keys with total order: numbers, strings, etc.

2. Data structure to store a subset $S \subseteq \mathcal{U}$

3. Operations:
   1. **Search/lookup**: given $x \in \mathcal{U}$ is $x \in S$?
   2. **Insert**: given $x \notin S$ add $x$ to $S$.
   3. **Delete**: given $x \in S$ delete $x$ from $S$

4. **Static** structure: $S$ given in advance or changes very infrequently, main operations are lookups.

5. **Dynamic** structure: $S$ changes rapidly so inserts and deletes as important as lookups.
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Common solutions:

1. **Static:**
   1. Store $S$ as a *sorted* array
   2. **Lookup:** Binary search in $O(\log |S|)$ time (comparisons)

2. **Dynamic:**
   1. Store $S$ in a *balanced* binary search tree
   2. Lookup, Insert, Delete in $O(\log |S|)$ time (comparisons)
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Question: “Should Tables be Sorted?”
(also title of famous paper by Turing award winner Andy Yao)

Hashing is a widely used & powerful technique for dictionaries.

Motivation:

1. Universe $\mathcal{U}$ may not be (naturally) totally ordered.
2. Keys correspond to large objects (images, graphs etc) for which comparisons are very expensive.
3. Want to improve “average” performance of lookups to $O(1)$ even at cost of extra space or errors with small probability: many applications for fast lookups in networking, security, etc.
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Hashing and Hash Tables

1. Hash Table data structure:
   1. A (hash) table/array $T$ of size $m$ (the table size).
   2. A hash function $h : \mathcal{U} \rightarrow \{0, \ldots, m - 1\}$.
   3. Item $x \in \mathcal{U}$ hashes to slot $h(x)$ in $T$.

2. Given $S \subseteq \mathcal{U}$. How do we store $S$ and how do we do lookups?

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**Ideal situation:**

1. Each element $x \in S$ hashes to a distinct slot in $T$. Store $x$ in slot $h(x)$

2. **Lookup:** Given $y \in \mathcal{U}$ check if $T[h(y)] = y$. $O(1)$ time!

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1. **Collision:** $h(x) = h(y)$ for some $x \neq y$.

2. **Chaining** to handle collisions:
   
   1. For each slot $i$ store all items hashed to slot $i$ in a linked list. $T[i]$ points to the linked list
   
   2. **Lookup:** to find if $y \in \mathcal{U}$ is in $T$, check the linked list at $T[h(y)]$. Time proportion to size of linked list.

3. This is also known as **Open hashing**.
Handling Collisions: Chaining

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   ![Diagram](attachment:image.png)

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   ![Diagram of a linked list for handling collisions](image)

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Handling Collisions

Several other techniques:

1. Open addressing.
   Every element has a list of places it can be (in certain order). Check in this order.

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3. Cuckoo hashing.
   Every value has two possible locations. When inserting, insert in one of the locations, otherwise, kick stored value to its other location. Repeat till stable. if no stability then rebuild table.

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Understanding Hashing

1. Does hashing give $O(1)$ time per operation for dictionaries?

2. Questions:
   1. Complexity of evaluating $h$ on a given element?
   2. Relative sizes of the universe $U$ and the set to be stored $S$.
   3. Size of table relative to size of $S$.
   4. Worst-case vs average-case vs randomized (expected) time?
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1. Considerations:
   1. Complexity of evaluating $h$ on a given element? Should be small.
   2. Relative sizes of the universe $\mathcal{U}$ and the set to be stored $S$: typically $|\mathcal{U}| \gg |S|$.
   3. Size of table relative to size of $S$. The load factor of $T$ is the ratio $n/t$ where $n = |S|$ and $m = |T|$. Typically $n/t$ is a small constant smaller than 1. Also known as the fill factor.

2. Main and interrelated questions:
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Main and interrelated questions:
1. Worst-case vs average-case vs randomized (expected) time?
2. How do we choose $h$?
Understanding Hashing

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Single hash function

1. \( \mathcal{U} \): universe (very large).

2. Assume \( N = |\mathcal{U}| \gg m \) where \( m \) is size of table \( T \). In particular assume \( N \geq m^2 \) (very conservative).

3. Fix hash function \( h : \mathcal{U} \rightarrow \{0, \ldots, m - 1\} \).

4. \( N \) items hashed to \( m \) slots. By pigeon hole principle there is some \( i \in \{0, \ldots, m - 1\} \) such that \( N/m \geq m \) elements of \( \mathcal{U} \) get hashed to \( i \) (!).

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Picking a hash function

1. How to pick functions?
   1. Hash functions are often chosen in an ad hoc fashion. Implicit assumption is that input behaves well.
   2. Theory and sound practice suggests that a hash function should be chosen properly with guarantees on its behavior.

2. Parameters: \( N = |U|, m = |T|, n = |S| \)
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OLD CS473
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1. **Question:** Why not let \( H \) be the set of all functions from \( U \) to \( \{0, 1, \ldots, m - 1\} \)?

2. **1** Too many functions! A random function has high complexity!
   - \# of functions: \( M = m^{|U|} \).
   - Bits to encode such a function \( \approx \log M = |U| \log m \).

3. **Question:** Are there good and compact families \( H \)?
   - Yes... But what it means for \( H \) to be good and compact.
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Question: What are good properties of $\mathcal{H}$ in distributing data?

1. Consider any element $x \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then $x$ should go into a random slot in $T$. In other words $\Pr[h(x) = i] = \frac{1}{m}$ for every $0 \leq i < m$.

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3. Second property is stronger than the first and the crucial issue.

Definition

A family hash function $\mathcal{H}$ is 2-universal if for all distinct $x, y \in \mathcal{U}$, $\Pr[h(x) = h(y)] = \frac{1}{m}$ where $m$ is the table size.

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1. $T$ is hash table of size $m$.
2. $S \subseteq U$ is a fixed set of size $\leq m$.
3. $h$ is chosen randomly from a uniform hash family $\mathcal{H}$.
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Analyzing Uniform Hashing

**Question:** What is the expected time to look up $x$ in $T$ using $h$ assuming chaining used to resolve collisions?

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Continued...

Number of elements colliding with $x$: $\ell(x) = \sum_{y \in S} D_y$.

$$\Rightarrow E[\ell(x)] = \sum_{y \in S} E[D_y] \quad \text{linearity of expectation}$$

$$= \sum_{y \in S} \Pr[h(x) = h(y)]$$

$$= \sum_{y \in S} \frac{1}{m} \quad \text{since } \mathcal{H} \text{ is a uniform hash family}$$

$$= \frac{|S|}{m}$$

$$\leq 1 \quad \text{if } |S| \leq m$$
Analyzing Uniform Hashing

1. **Question**: What is the *expected* time to look up $x$ in $T$ using $h$ assuming chaining used to resolve collisions?

2. **Answer**: $O(n/m)$.

3. **Comments**:
   1. $O(1)$ expected time also holds for insertion.
   2. Analysis assumes static set $S$ but holds as long as $S$ is a set formed with at most $O(m)$ insertions and deletions.
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... making the hash table dynamic

Previous analysis assumed fixed $S$ of size $\sim m$.

**Question:** What happens as items are inserted and deleted?

1. If $|S|$ grows to more than $cm$ for some constant $c$ then hash table performance clearly degrades.

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**Solution:** Rebuild hash table periodically!

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1. Start with table size $m$ where $m$ is some estimate of $|S|$ (can be some large constant).

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3. If $|S|$ stays roughly the same but more than $c|S|$ operations on table for some chosen constant $c$ (say 10), rebuild.

The amortize cost of rebuilding to previously performed operations. Rebuilding ensures $O(1)$ expected analysis holds even when $S$ changes. Hence $O(1)$ expected look up/insert/delete time dynamic data dictionary data structure!
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Lemma

Let $p$ be a prime number,

$x$: an integer number in $\{1, \ldots, p - 1\}$.

There exists a unique $y$ s.t. $xy = 1 \mod p$.

In other words: For every element there is a unique inverse.

$\mathbb{Z}_p = \{0, 1, \ldots, p - 1\}$ when working module $p$ is a field.
Proof of lemma

Claim

Let $p$ be a prime number. For any $\alpha, \beta, i \in \{1, \ldots, p - 1\}$ s.t. $\alpha \neq \beta$, we have that $\alpha i \neq \beta i \mod p$.

Proof.

Assume for the sake of contradiction $\alpha i = \beta i \mod p$. Then

$$i(\alpha - \beta) = 0 \mod p$$

$\implies p$ divides $i(\alpha - \beta)$

$\implies p$ divides $\alpha - \beta$

$\implies \alpha - \beta = 0$

$\implies \alpha = \beta$.

And that is a contradiction.
Proof of lemma...

Uniqueness.

**Lemma**

Let $p$ be a prime number, $x$: an integer number in $\{1, \ldots, p-1\}$.

$\implies$ There exists a unique $y$ s.t. $xy = 1 \mod p$.

**Proof.**

Assume the lemma is false and there are two distinct numbers $y, z \in \{1, \ldots, p-1\}$ such that

$$xy = 1 \mod p \quad \text{and} \quad xz = 1 \mod p.$$

But this contradicts the above claim (set $i = x$, $\alpha = y$ and $\beta = z$).
Proof of lemma...

Existence

Proof.

By claim, for any $\alpha \in \{1, \ldots, p - 1\}$ we have that
$$\{\alpha \times 1 \mod p, \alpha \times 2 \mod p, \ldots, \alpha \times (p - 1) \mod p\} = \{1, 2, \ldots, p - 1\}.$$  

Therefore, there exists a number $y \in \{1, \ldots, p - 1\}$ such that $\alpha y = 1 \mod p$.  

Constructing Universal Hash Families

Parameters: \( N = |\mathcal{U}|, m = |\mathcal{T}|, n = |\mathcal{S}| \)

1. Choose a prime number \( p \geq N \). \( \mathbb{Z}_p = \{0, 1, \ldots, p - 1\} \) is a field.
2. For \( a, b \in \mathbb{Z}_p, a \neq 0 \), define the hash function \( h_{a,b} \) as
   \[ h_{a,b}(x) = ((ax + b) \mod p) \mod m. \]
3. Let \( \mathcal{H} = \{ h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0 \} \). Note that
   \( |\mathcal{H}| = p(p - 1) \).

Theorem

\( \mathcal{H} \) is a 2-universal hash family.

Comments:

1. Hash family is of small size, easy to sample from.
2. Easy to store a hash function (\( a, b \) have to be stored) and evaluate it.
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What the is going on?

\[ h_{a,b}(x) = ((ax + b) \mod p) \mod m \]

First map \( x \neq y \) to \( r = h(x) \) and \( s = h(y) \).

This is a random uniform mapping (choosing \( a \) and \( b \)) – every cell has the same probability to be the target, for fixed \( x \) and \( y \).
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1. First part of mapping maps \((x, y)\) to a random location \((h_{a,b}(x), h_{a,b}(y))\) in the “matrix”.

2. \((h_{a,b}(x), h_{a,b}(y))\) is not on main diagonal.

3. All blue locations are “bad” – map by \(\mod m\) to a location of collusion.

4. But… at most \(1/m\) fraction of allowable locations in the matrix are bad.
Constructing Universal Hash Families

**Theorem**

\( \mathcal{H} \) is a (2)-universal hash family.

**Proof.**

Fix \( x, y \in \mathcal{U} \). What is the probability they will collide if \( h \) is picked randomly from \( \mathcal{H} \)?

1. Let \( a, b \) be bad for \( x, y \) if \( h_{a,b}(x) = h_{a,b}(y) \).
2. **Claim:** Number of bad pairs is at most \( p(p - 1)/m \).
3. Total number of hash functions is \( p(p - 1) \) and hence probability of a collision is \( \leq 1/m \).
Theorem

$\mathcal{H}$ is a (2)-universal hash family.

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\[\Box\]
Lemma

If $x \neq y$ then for any $a, b \in \mathbb{Z}_p$ such that $a \neq 0$, we have

$$ax + b \mod p \neq ay + b \mod p.$$  

Proof.

If $ax + b \mod p = ay + b \mod p$ then $a(x - y) \mod p = 0$ and $a \neq 0$ and $(x - y) \neq 0$. However, $a$ and $(x - y)$ cannot divide $p$ since $p$ is prime and $a < p$ and $(x - y) < p$.  

☐
Lemma

If \( x \neq y \) then for each \((r, s)\) such that \( r \neq s \) and \( 0 \leq r, s \leq p - 1 \) there is exactly one \( a, b \) such that
\[
ax + b \mod p = r \quad \text{and} \quad ay + b \mod p = s.
\]

Proof.

Solve the two equations:
\[
ax + b = r \mod p \quad \text{and} \quad ay + b = s \mod p
\]

We get \( a = \frac{r-s}{x-y} \mod p \) and \( b = r - ax \mod p \).
Understanding the hashing

Once we fix $a$ and $b$, and we are given a value $x$, we compute the hash value of $x$ in two stages:

1. **Compute**: $r \leftarrow (ax + b) \mod p$.
2. **Fold**: $r' \leftarrow r \mod m$

Collision...

Given two values $x$ and $y$ they might collide because of either steps.

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$\# \text{ not equal pairs of } \mathbb{Z}_p \times \mathbb{Z}_p \text{ that are folded to the same number is } p(p - 1)/m.$
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Consider a pair $(x, y) \in \{0, 1, \ldots, p - 1\}^2$ s.t. $x \neq y$. Fix $x$:

1. There are $\lceil p/m \rceil$ values of $y$ that fold into $x$. That is
   
   \[ x \mod m = y \mod m. \]

2. One of them is when $x = y$.

3. \# of colliding pairs $(\lceil p/m \rceil - 1)p \leq (p - 1)p/m$
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Hashing used typically for integers, vectors, strings etc.

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