Greedy Algorithms for Minimum Spanning Trees

Lecture 13
March 5, 2015

Minimum Spanning Tree

**Input** Connected graph \( G = (V, E) \) with edge costs

**Goal** Find \( T \subseteq E \) such that \( (V, T) \) is connected and total cost of all edges in \( T \) is smallest

- \( T \) is the **minimum spanning tree** (MST) of \( G \)

Applications

1. **Network Design**
   - Designing networks with minimum cost but maximum connectivity

2. **Approximation algorithms**
   - Can be used to bound the optimality of algorithms to approximate Traveling Salesman Problem, Steiner Trees, etc.

3. **Cluster Analysis**
Greedy Template

Initially $E$ is the set of all edges in $G$.

1. $T$ is empty (i.e., $T$ will store edges of a MST).
2. While $E$ is not empty do
   a. Choose $i \in E$.
   b. If $i$ satisfies condition, add $i$ to $T$.
3. Return the set $T$.

Main Task: In what order should edges be processed? When should we add an edge to spanning tree?

Prim’s Algorithm

$T$ maintained by algorithm will be a tree. Start with a node in $T$. In each iteration, pick edge with least attachment cost to $T$.

Kruskal’s Algorithm

Process edges in the order of their costs (starting from the least) and add edges to $T$ as long as they don’t form a cycle.

Reverse Delete Algorithm

Initially $E$ is the set of all edges in $G$.

1. $T$ is $E$ (i.e., $T$ will store edges of a MST).
2. While $E$ is not empty do
   a. Choose $i \in E$ of largest cost.
   b. If removing $i$ does not disconnect $T$ then
      i. Remove $i$ from $T$.
3. Return the set $T$.

Returns a minimum spanning tree.
Correctness of MST Algorithms

- Many different MST algorithms
- All of them rely on some basic properties of MSTs, in particular the Cut Property to be seen shortly.

Assumption

And for now ...

Assumption

Edge costs are distinct, that is no two edge costs are equal.

Cuts

Definition

- \( G = (V, E) \): graph. A cut is a partition of the vertices of the graph into two sets \((S, V \setminus S)\).
- Edges having an endpoint on both sides are the edges of the cut.
- A cut edge is crossing the cut.

Safe and Unsafe Edges

Definition

- An edge \( e = (u, v) \) is a safe edge if there is some partition of \( V \) into \( S \) and \( V \setminus S \) and \( e \) is the unique minimum cost edge crossing \( S \) (one end in \( S \) and the other in \( V \setminus S \)).

Definition

- An edge \( e = (u, v) \) is an unsafe edge if there is some cycle \( C \) such that \( e \) is the unique maximum cost edge in \( C \).

Proposition

If edge costs are distinct then every edge is either safe or unsafe.

Proof.

Exercise.
Safe edge
Example...

- Every cut identifies one safe edge...

...the cheapest edge in the cut.
- Note: An edge $e$ may be a safe edge for many cuts!

Unsafe edge
Example...

- Every cycle identifies one unsafe edge...

...the most expensive edge in the cycle.

Example

![Graph with unique edge costs. Safe edges are red, rest are unsafe.]

And all safe edges are in the MST in this case...

Key Observation: Cut Property

**Lemma**

If $e$ is a safe edge then every minimum spanning tree contains $e$.

**Proof.**

- Suppose (for contradiction) $e$ is not in MST $T$.
- Since $e$ is safe there is an $S \subset V$ such that $e$ is the unique min cost edge crossing $S$.
- Since $T$ is connected, there must be some edge $f$ with one end in $S$ and the other in $V \setminus S$.
- Since $c_f > c_e$, $T' = (T \setminus \{f\}) \cup \{e\}$ is a spanning tree of lower cost!
- **Error:** $T'$ may not be a spanning tree!!
Proof of Cut Property (contd)

Observation

\[ T' = (T \setminus \{e'\}) \cup \{e\} \] is a spanning tree.

Proof.

\( T' \) is connected.

Removed \( e' = (v', w') \) from \( T \) but \( v' \) and \( w' \) are connected by the path \( P - f + e \) in \( T' \). Hence \( T' \) is connected if \( T \) is.

\( T' \) is a tree

\( T' \) is connected and has \( n - 1 \) edges (since \( T \) had \( n - 1 \) edges) and hence \( T' \) is a tree.

Proof of Cut Property

Suppose \( e = (v, w) \) is not in \( MST \).

\( T \) and \( e \) is min weight edge in cut \((S, V \setminus S)\).

Let \( v \in S \).

\( T \) is spanning tree: there is a unique path \( P \) from \( v \) to \( w \) in \( T \).

Let \( w' \) be the first vertex in \( P \) belonging to \( V \setminus S \); let \( v' \) be the vertex just before it on \( P \), and let \( e' = (v', w') \).

\[ T' = (T \setminus \{e'\}) \cup \{e\} \] is spanning tree of lower cost. (Why?)

Safe Edges form a Tree

Lemma

Let \( G \) be a connected graph with distinct edge costs, then the set of safe edges form a connected graph.

Proof.

Suppose not. Let \( S \) be a connected component in the graph induced by the safe edges.

Consider the edges crossing \( S \), there must be a safe edge among them since edge costs are distinct and so we must have picked it.
Safe Edges form an MST

**Corollary**

Let $G$ be a connected graph with distinct edge costs, then set of safe edges form the unique MST of $G$.

**Consequence:** Every correct MST algorithm when $G$ has unique edge costs includes exactly the safe edges.

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**Cycle Property**

**Lemma**

If $e$ is an unsafe edge then no MST of $G$ contains $e$.

**Proof.**

Exercise. See textbook.

**Note:** Cut and Cycle properties hold even when edge costs are not distinct. Safe and unsafe definitions do not rely on distinct cost assumption.

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**Correctness of Prim’s Algorithm**

**Prim’s Algorithm**

Pick edge with minimum attachment cost to current tree, and add to current tree.

**Proof of correctness.**

1. If $e$ is added to tree, then $e$ is safe and belongs to every MST.
   - Let $S$ be the vertices connected by edges in $T$ when $e$ is added.
   - $e$ is edge of lowest cost with one end in $S$ and the other in $V \setminus S$ and hence $e$ is safe.
2. Set of edges output is a spanning tree
   - Set of edges output forms a connected graph: by induction, $S$ is connected in each iteration and eventually $S = V$.
   - Only safe edges added and they do not have a cycle

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**Correctness of Kruskal’s Algorithm**

**Kruskal’s Algorithm**

Pick edge of lowest cost and add if it does not form a cycle with existing edges.

**Proof of correctness.**

1. If $e = (u, v)$ is added to tree, then $e$ is safe
   - When algorithm adds $e$ let $S$ and $S'$ be the connected components containing $u$ and $v$ respectively
   - $e$ is the lowest cost edge crossing $S$ (and also $S'$).
   - If there is an edge $e'$ crossing $S$ and has lower cost than $e$, then $e'$ would come before $e$ in the sorted order and would be added by the algorithm to $T$
2. Set of edges output is a spanning tree: exercise
Correctness of Reverse Delete Algorithm

Reverse Delete Algorithm
Consider edges in decreasing cost and remove an edge if it does not disconnect the graph

Proof of correctness.
Argue that only unsafe edges are removed (see text book).

When edge costs are not distinct

Heuristic argument: Make edge costs distinct by adding a small tiny and different cost to each edge

Formal argument: Order edges lexicographically to break ties
1. \( e_i < e_j \) if either \( c(e_i) < c(e_j) \) or \( c(e_i) = c(e_j) \) and \( i < j \)
2. Lexicographic ordering extends to sets of edges. If \( A, B \subseteq E \), \( A \neq B \) then \( A < B \) if either \( c(A) < c(B) \) or \( c(A) = c(B) \) and \( A \setminus B \) has a lower indexed edge than \( B \setminus A \)
3. Can order all spanning trees according to lexicographic order of their edge sets. Hence there is a unique MST.

Prim’s, Kruskal, and Reverse Delete Algorithms are optimal with respect to lexicographic ordering.

Edge Costs: Positive and Negative

1. Algorithms and proofs don’t assume that edge costs are non-negative! MST algorithms work for arbitrary edge costs.
2. Another way to see this: make edge costs non-negative by adding to each edge a large enough positive number. Why does this work for MSTs but not for shortest paths?
3. Can compute maximum weight spanning tree by negating edge costs and then computing an MST.

Part II
Data Structures for MST: Priority Queues and Union-Find
Implementing Prim’s Algorithm

Prim_ComputeMST

\[E\] is the set of all edges in \(G\)
\[S = \{1\}\]
\(T\) is empty (* \(T\) will store edges of a MST *)

while \(S \neq V\) do
    \[e = (v, w) \in E\] such that
    \[v \in S\] and \(w \in V - S\)
    \(e\) has minimum cost
    \[T = T \cup e\]
    \(S = S \cup w\)

return the set \(T\)

Analysis

- Number of iterations = \(O(n)\), where \(n\) is number of vertices
- Picking \(e\) is \(O(m)\) where \(m\) is the number of edges
- Total time \(O(nm)\)

More Efficient Implementation

Prim_ComputeMST

\[E\] is the set of all edges in \(G\)
\[S = \{1\}\]
\(T\) is empty (* \(T\) will store edges of a MST *)

for \(v \not\in S\), \(a(v) = \min_{w \in S} c(w, v)\)
for \(v \not\in S\), \(e(v) = w\) such that \(w \in S\) and \(c(w, v)\) is minimum

while \(S \neq V\) do
    pick \(v\) with minimum \(a(v)\)
    \[T = T \cup \{(e(v), v)\}\]
    \(S = S \cup \{v\}\)
    update arrays \(a\) and \(e\)

return the set \(T\)

Maintain vertices in \(V \setminus S\) in a priority queue with key \(a(v)\).

Priority Queues

Data structure to store a set \(S\) of \(n\) elements where each element \(v \in S\) has an associated real/integer key \(k(v)\) such that the following operations

- \(\text{makeQ}\): create an empty queue
- \(\text{findMin}\): find the minimum key in \(S\)
- \(\text{extractMin}\): Remove \(v \in S\) with smallest key and return it
- \(\text{add}(v, k(v))\): Add new element \(v\) with key \(k(v)\) to \(S\)
- \(\text{Delete}(v)\): Remove element \(v\) from \(S\)
- \(\text{decreaseKey}(v, k'(v))\): decrease key of \(v\) from \(k(v)\) (current key) to \(k'(v)\) (new key). Assumption: \(k'(v) \leq k(v)\)
- \(\text{meld}\): merge two separate priority queues into one

Prim’s using priority queues

\[E\] is the set of all edges in \(G\)
\[S = \{1\}\]
\(T\) is empty (* \(T\) will store edges of a MST *)

for \(v \not\in S\), \(a(v) = \min_{w \in S} c(w, v)\)
for \(v \not\in S\), \(e(v) = w\) such that \(w \in S\) and \(c(w, v)\) is minimum

while \(S \neq V\) do
    pick \(v\) with minimum \(a(v)\)
    \[T = T \cup \{(e(v), v)\}\]
    \(S = S \cup \{v\}\)
    update arrays \(a\) and \(e\)

return the set \(T\)

Maintain vertices in \(V \setminus S\) in a priority queue with key \(a(v)\)

- Requires \(O(n)\) \(\text{extractMin}\) operations
- Requires \(O(m)\) \(\text{decreaseKey}\) operations
Running time of Prim’s Algorithm

\( O(n) \) \text{ \texttt{extractMin} operations and } \( O(m) \) \text{ \texttt{decreaseKey} operations }

- Using standard Heaps, \( \text{\texttt{extractMin}} \) and \( \text{\texttt{decreaseKey}} \) take \( O(\log n) \) time. Total: \( O((m + n) \log n) \)
- Using Fibonacci Heaps, \( O(\log n) \) for \( \text{\texttt{extractMin}} \) and \( O(1) \) (amortized) for \( \text{\texttt{decreaseKey}} \). Total: \( O(n \log n + m) \).

Prim’s algorithm and Dijkstra’s algorithms are similar. Where is the difference?

Kruskal’s Algorithm

\textbf{Kruskal\_ComputeMST}

- Initially \( E \) is the set of all edges in \( G \)
- \( T \) is empty (\( \ast T \) will store edges of a MST \( \ast \))
- while \( E \) is not empty do
  - choose \( e \in E \) of minimum cost
  - if \((T \cup \{e\}) \) does not have cycles
    - add \( e \) to \( T \)
  - return the set \( T \)

1. Presort edges based on cost. Choosing minimum can be done in \( O(1) \) time
2. Do BFS/DFS on \( T \cup \{e\} \). Takes \( O(n) \) time
3. Total time \( O(m \log m) + O(mn) = O(mn) \)

Implementing Kruskal’s Algorithm Efficiently

\textbf{Kruskal\_ComputeMST}

- Sort edges in \( E \) based on cost
- \( T \) is empty (\( \ast T \) will store edges of a MST \( \ast \))
- each vertex \( u \) is placed in a set by itself
- while \( E \) is not empty do
  - pick \( e = (u, v) \in E \) of minimum cost
  - if \( u \) and \( v \) belong to different sets
    - add \( e \) to \( T \)
  - merge the sets containing \( u \) and \( v \)
- return the set \( T \)

Need a data structure to check if two elements belong to same set and to merge two sets.

Union-Find Data Structure

\textbf{Data Structure}

Store disjoint sets of elements that supports the following operations
1. \textbf{makeUnionFind} \((S)\) returns a data structure where each element of \( S \) is in a separate set
2. \textbf{find} \((u)\) returns the \textit{name} of set containing element \( u \). Thus, \( u \) and \( v \) belong to the same set if and only if \( \text{\texttt{find}}(u) = \text{\texttt{find}}(v) \)
3. \textbf{union} \((A, B)\) merges two sets \( A \) and \( B \). Here \( A \) and \( B \) are the names of the sets. Typically the name of a set is some element in the set.
Implementing Union-Find using Arrays and Lists

Using lists
- Each set stored as list with a name associated with the list.
- For each element \( u \in S \) a pointer to its set. Array for pointers: \( \text{component} \left[ u \right] \) is pointer for \( u \).
- \( \text{makeUnionFind} \left( S \right) \) takes \( O(n) \) time and space.

Example

```
  s
  t
  u
  w
  y
  v
  x
  y
  z
```

Implementing Union-Find using Arrays and Lists

- \( \text{find} \left( u \right) \) reads the entry \( \text{component} \left[ u \right] \): \( O(1) \) time
- \( \text{union} \left( A, B \right) \) involves updating the entries \( \text{component} \left[ u \right] \) for all elements \( u \) in \( A \) and \( B \): \( O(|A| + |B|) \) which is \( O(n) \)

```
s t
s t
u w y
u w y
v x
v x u w y
x
y
Z
Z
```

Improving the List Implementation for Union

New Implementation
As before use \( \text{component} \left[ u \right] \) to store set of \( u \).

- with each set, keep track of its size
- assume \( |A| \leq |B| \) for now
- Merge the list of \( A \) into that of \( B \): \( O(1) \) time (linked lists)
- Update \( \text{component} \left[ u \right] \) only for elements in the smaller set \( A \)
- Total \( O(|A|) \) time. Worst case is still \( O(n) \).

\( \text{find} \) still takes \( O(1) \) time
Example

```
  s  t
 u  w  y
 v  x
 w  y  x
 x  y
 z
```

The smaller set (list) is appended to the largest set (list)

Improving the List Implementation for Union

**Question**

Is the improved implementation provably better or is it simply a nice heuristic?

**Theorem**

Any sequence of \( k \) union operations, starting from \( \text{makeUnionFind}(S) \) on set \( S \) of size \( n \), takes at most \( O(k \log k) \).

**Corollary**

Kruskal’s algorithm can be implemented in \( O(m \log m) \) time.

**Proof**

Any union operation involves at most 2 of the original one-element sets; thus at least \( n - 2k \) elements have never been involved in a union.

Also, maximum size of any set (after \( k \) unions) is \( 2k \).

**Key Observation**

\( \text{union}(A, B) \) takes \( O(|A|) \) time where \( |A| \leq |B| \). Size of new set is \( \geq 2|A| \). Cannot double too many times.

Proof.

- Any union operation involves at most 2 of the original one-element sets; thus at least \( n - 2k \) elements have never been involved in a union.
- Also, maximum size of any set (after \( k \) unions) is \( 2k \).
- \( \text{union}(A, B) \) takes \( O(|A|) \) time where \( |A| \leq |B| \).
- Charge each element in \( A \) constant time to pay for \( O(|A|) \) time.
- How much does any element get charged?
- If component \([v]\) is updated, set containing \( v \) doubles in size.
- component \([v]\) is updated at most \( \log 2k \) times.
- Total number of updates is \( 2k \log 2k = O(k \log k) \).
Improving Worst Case Time

Better data structure
Maintain elements in a forest of in-trees; all elements in one tree belong to a set with root’s name.

- **find**(u): Traverse from u to the root
- **union**(A, B): Make root of A (smaller set) point to root of B.
  Takes O(1) time.

Analysis

**Theorem**
The forest based implementation for a set of size n, has the following complexity for the various operations: makeUnionFind takes O(n), union takes O(1), and find takes O(\(\log n\)).

**Proof.**

- **find**(u) depends on the height of tree containing u.
- Height of u increases by at most 1 only when the set containing u changes its name.
- If height of u increases then size of the set containing u (at least) doubles.
- Maximum set size is n; so height of any tree is at most O(\(\log n\)).

Details of Implementation

Each element \(u \in S\) has a pointer parent(\(u\)) to its ancestor.

```plaintext
makeUnionFind(S)
    for each u in S do
        parent(u) = u

find(u)
    while (parent(u) \neq u) do
        u = parent(u)
    return u
```

```plaintext
union(component(u), component(v))
    (* parent(u) = u & parent(v) = v *)
    if (|component(u)| \leq |component(v)|) then
        parent(u) = v
    else
        parent(v) = u
    set new component size to |component(u)| + |component(v)|
```

Further Improvements: Path Compression

**Observation**
Consecutive calls of **find**(u) take \(O(\log n)\) time each, but they traverse the same sequence of pointers.

**Idea: Path Compression**
Make all nodes encountered in the **find**(u) point to root.
Path Compression: Example

Path Compression

```plaintext
find(u):
  if (parent(u) ≠ u) then
    parent(u) = find(parent(u))
  return parent(u)
```

Question

Does Path Compression help?

Yes!

Theorem

With Path Compression, \( k \) operations (find and/or union) take \( O(k\alpha(k, \min\{k, n\})) \) time where \( \alpha \) is the inverse Ackermann function.

Acknowledgement and Inverse Ackermann Functions

Ackermann function \( A(m, n) \) defined for \( m, n ≥ 0 \) recursively

\[
A(m, n) = \begin{cases} 
  n + 1 & \text{if } m = 0 \\
  A(m - 1, 1) & \text{if } m > 0 \text{ and } n = 0 \\
  A(m - 1, A(m, n - 1)) & \text{if } m > 0 \text{ and } n > 0
\end{cases}
\]

\( A(3, n) = 2^{n+3} - 3 \)
\( A(4, 3) = 2^{65536} - 3 \)

\( \alpha(m, n) \) is inverse Ackermann function defined as

\[
\alpha(m, n) = \min\{i \mid A(i, \lfloor m/n \rfloor) ≥ \log_2 n\}
\]

For all practical purposes \( \alpha(m, n) ≤ 5 \)

Lower Bound for Union-Find Data Structure

Amazing result:

Theorem (Tarjan)

For Union-Find, any data structure in the pointer model requires \( \Omega(m\alpha(m, n)) \) time for \( m \) operations.
Running time of Kruskal’s Algorithm

Using Union-Find data structure:

1. \( O(m) \) \textbf{find} operations (two for each edge)
2. \( O(n) \) \textbf{union} operations (one for each edge added to \( T \))
3. Total time: \( O(m \log m) \) for sorting plus \( O(m \alpha(n)) \) for union-find operations. Thus \( O(m \log m) \) time despite the improved Union-Find data structure.

Best Known Asymptotic Running Times for MST

Prim’s algorithm using Fibonacci heaps: \( O(n \log n + m) \).

If \( m \) is \( O(n) \) then running time is \( \Omega(n \log n) \).

Question

Is there a linear time \( (O(m + n) \) time) algorithm for MST?

1. \( O(m \log^* m) \) time Fredman and Tarjan [1987].
2. \( O(m + n) \) time using bit operations in RAM model Fredman and Willard [1994].
3. \( O(m + n) \) expected time (randomized algorithm) Karger et al. [1995].
4. \( O((n + m) \alpha(m, n)) \) time Chazelle [2000].
5. Still open: Is there an \( O(n + m) \) time deterministic algorithm in the comparison model?


