Chapter 4

Breadth First Search, Dijkstra’s Algorithm for Shortest Paths

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4.1 Breadth First Search
4.1.0.1 Breadth First Search (BFS)

Overview

(A) BFS is obtained from BasicSearch by processing edges using a queue data structure.

(B) It processes the vertices in the graph in the order of their shortest distance from the vertex s (the start vertex).

As such...

(A) DFS good for exploring graph structure
(B) BFS good for exploring distances

4.1.0.2 Queue Data Structure

Queues

queue: list of elements which supports the operations:
(A) enqueue: Adds an element to the end of the list
(B) dequeue: Removes an element from the front of the list
Elements are extracted in first-in first-out (FIFO) order, i.e., elements are picked in the order in which they were inserted.
4.1.0.3 BFS Algorithm

Given (undirected or directed) graph $G = (V, E)$ and node $s \in V$

\[
\text{BFS}(s)
\]
Mark all vertices as unvisited
Initialize search tree $T$ to be empty
Mark vertex $s$ as visited
set $Q$ to be the empty queue
\[\text{enq}(s)\]
while $Q$ is nonempty do
\[u = \text{deq}(Q)\]
for each vertex $v \in \text{Adj}(u)$
if $v$ is not visited then
add edge $(u, v)$ to $T$
Mark $v$ as visited and \[\text{enq}(v)\]

Proposition 4.1.1. \text{BFS}(s) runs in $O(n + m)$ time.

4.1.0.4 BFS: An Example in Undirected Graphs

BFS tree is the set of black edges.

BFS

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4.1.0.5 BFS: An Example in Directed Graphs

4.1.0.6 BFS with Distance

**BFS**($s$)
Mark all vertices as unvisited and for each $v$ set dist($v$) = \infty
Initialize search tree $T$ to be empty
Mark vertex $s$ as visited and set dist($s$) = 0
set $Q$ to be the empty queue
enq($s$)
while $Q$ is nonempty do
  $u = \text{deq}(Q)$
  for each vertex $v \in \text{Adj}(u)$ do
    if $v$ is not visited do
      add edge $(u, v)$ to $T$
      Mark $v$ as visited, enq($v$)
      and set dist($v$) = dist($u$) + 1

4.1.0.7 Properties of BFS: Undirected Graphs

**Proposition 4.1.2.** The following properties hold upon termination of **BFS**($s$):
(A) $\forall (\text{BFS tree comp.}) =$ set vertices in connected component $s$.
(B) If dist($u$) < dist($v$) then $u$ is visited before $v$.
(C) $\forall u \in V$, dist($u$) = the length of shortest path from $s$ to $u$.
(D) If $u, v \in$ connected component of $s$, and $e = uv$ is an edge of $G$, then either $e \in$ BFS tree, or $|\text{dist}(u) - \text{dist}(v)| \leq 1$.

**Proof**: Exercise.

4.1.0.8 Properties of BFS: Directed Graphs

**Proposition 4.1.3.** The following properties hold upon termination of $T \leftarrow \text{BFS}(s)$:
(A) For search tree $T$. $\forall (T) =$ set of vertices reachable from $s$
(B) If dist($u$) < dist($v$) then $u$ is visited before $v$
(C) $\forall u \in V(T)$: dist($u$) = length of shortest path from $s$ to $u$
(D) If $u$ is reachable from $s$, $e = (u \rightarrow v) \in E(G)$.
  Then either (i) $e$ is an edge in the search tree,
  or (ii) dist($v$) − dist($u$) ≤ 1.

*Not necessarily the case that* dist($u$) − dist($v$) ≤ 1.
Proof: Exercise.

4.1.0.9 BFS with Layers

**BFSLayers\(s\):**

Mark all vertices as unvisited and initialize \(T\) to be empty
Mark \(s\) as visited and set \(L_0 = \{s\}\)
\(i = 0\)
while \(L_i\) is not empty do
  initialize \(L_{i+1}\) to be an empty list
  for each \(u\) in \(L_i\) do
    for each edge \((u, v) \in \text{Adj}(u)\) do
      if \(v\) is not visited
        mark \(v\) as visited
        add \((u, v)\) to tree \(T\)
        add \(v\) to \(L_{i+1}\)
  \(i = i + 1\)

Running time: \(O(n + m)\)

4.1.0.10 Example

![Diagram showing BFS with Layers](image)

4.1.0.11 BFS with Layers: Properties

**Proposition 4.1.4.** The following properties hold on termination of \(\text{BFSLayers}(s)\).

(A) \(\text{BFSLayers}(s)\) outputs a BFS tree
(B) \(L_i\) is the set of vertices at distance exactly \(i\) from \(s\)
(C) If \(G\) is undirected, each edge \(e = uv\) is one of three types:
    (A) tree edge between two consecutive layers
    (B) non-tree **forward/backward** edge between two consecutive layers
    (C) non-tree **cross-edge** with both \(u, v\) in same layer
(D) \(\implies\) Every edge in the graph is either between two vertices that are either (i) in the same layer, or (ii) in two consecutive layers.
4.1.0.12 Example: Tree/cross/forward (backward) edges

4.1.1 BFS with Layers: Properties

4.1.1.1 For directed graphs

**Proposition 4.1.5.** The following properties hold on termination of $\text{BFS}^{\text{Layers}}(s)$, if $G$ is directed.

For each edge $e = (u \rightarrow v)$ is one of four types:

(A) a **tree** edge between consecutive layers, $u \in L_i, v \in L_{i+1}$ for some $i \geq 0$

(B) a non-tree **forward** edge between consecutive layers

(C) a non-tree **backward** edge

(D) a **cross-edge** with both $u, v$ in same layer

4.2 Bipartite Graphs and an application of BFS

4.2.0.2 Bipartite Graphs

**Definition 4.2.1 (Bipartite Graph).** Undirected graph $G = (V, E)$ is a **bipartite graph** if $V$ can be partitioned into $X$ and $Y$ s.t. all edges in $E$ are between $X$ and $Y$. 
4.2.0.3 Bipartite Graph Characterization

Question When is a graph bipartite?

**Proposition 4.2.2.** Every tree is a bipartite graph.

*Proof:* Root tree $T$ at some node $r$. Let $L_i$ be all nodes at level $i$, that is, $L_i$ is all nodes at distance $i$ from root $r$. Now define $X$ to be all nodes at even levels and $Y$ to be all nodes at odd level. Only edges in $T$ are between levels.

**Proposition 4.2.3.** An odd length cycle is not bipartite.

4.2.0.4 Odd Cycles are not Bipartite

**Proposition 4.2.4.** An odd length cycle is not bipartite.

*Proof:* Let $C = u_1, u_2, \ldots, u_{2k+1}, u_1$ be an odd cycle. Suppose $C$ is a bipartite graph and let $X, Y$ be the partition. Without loss of generality $u_1 \in X$. Implies $u_2 \in Y$. Implies $u_3 \in X$. Inductively, $u_i \in X$ if $i$ is odd $u_i \in Y$ if $i$ is even. But $\{u_1, u_{2k+1}\}$ is an edge and both belong to $X$.

4.2.0.5 Subgraphs

**Definition 4.2.5.** Given a graph $G = (V, E)$ a **subgraph** of $G$ is another graph $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$.

**Proposition 4.2.6.** If an undirected $G$ is bipartite then any subgraph $H$ of $G$ is also bipartite.

**Proposition 4.2.7.** An undirected graph $G$ is not bipartite if $G$ has an odd cycle $C$ as a subgraph.

*Proof:* If $G$ is bipartite then since $C$ is a subgraph, $C$ is also bipartite (by above proposition). However, $C$ is not bipartite!

4.2.0.6 Bipartite Graph Characterization

**Theorem 4.2.8.** An undirected graph $G$ is bipartite $\iff$ it has no odd length cycle as subgraph.

*Proof:* **Only If:** $G$ has an odd cycle implies $G$ is not bipartite. 

**If:** $G$ has no odd length cycle. Assume without loss of generality that $G$ is connected. 

(A) Pick $u$ arbitrarily and do $\text{BFS}(u)$

(B) $X = \bigcup_{i \text{ is even}} L_i$ and $Y = \bigcup_{i \text{ is odd}} L_i$

(C) **Claim:** $X$ and $Y$ is a valid partition if $G$ has no odd length cycle.
4.2.0.7 Proof of Claim

Claim 4.2.9. In BFS\((u)\) if \(a, b \in L_i\) and \(ab \in E(G)\) then there is an odd length cycle containing \(ab\).

Proof: Let \(v\) be least common ancestor of \(a, b\) in BFS tree \(T\).
\(v\) is in some level \(j < i\) (could be \(u\) itself).
Path from \(v \leadsto a\) in \(T\) is of length \(j - i\).
Path from \(v \leadsto b\) in \(T\) is of length \(j - i\).
These two paths plus \((a, b)\) forms an odd cycle of length \(2(j - i) + 1\).

4.2.0.8 Proof of Claim: Figure

4.2.0.9 Another tidbit

Corollary 4.2.10. There is an \(O(n+m)\) time algorithm to check if \(G\) is bipartite and output an odd cycle if it is not.

4.3 Shortest Paths and Dijkstra’s Algorithm

4.3.0.10 Shortest Path Problems

Input A (undirected or directed) graph \(G = (V, E)\) with edge lengths (or costs). For edge \(e = (u \rightarrow v)\), \(\ell(e) = \ell(u \rightarrow v)\) is its length.

(A) Given nodes \(s, t\) find shortest path from \(s\) to \(t\).
(B) Given node \(s\) find shortest path from \(s\) to all other nodes.
(C) Find shortest paths for all pairs of nodes.
   Many applications!

4.3.1 Single-Source Shortest Paths:

4.3.1.1 Non-Negative Edge Lengths

Single-Source Shortest Path Problems
(A) Input: A (undirected or directed) graph \(G = (V, E)\) with non-negative edge lengths.
   For edge \(e = (u \rightarrow v)\), \(\ell(e) = \ell(u \rightarrow v)\) is its length.
(B) Given nodes \(s, t\) find shortest path from \(s\) to \(t\).
(C) Given node \(s\) find shortest path from \(s\) to all other nodes.
(A) Restrict attention to directed graphs
(B) Undirected graph problem can be reduced to directed graph problem - how?
   (A) Given undirected graph \(G\), create a new directed graph \(G'\) by replacing each edge \(\{u, v\}\) in \(G\) by \((u \rightarrow v)\) and \((v, u)\) in \(G'\).
   (B) set \(\ell(u \rightarrow v) = \ell(v, u) = \ell(\{u, v\})\)
   (C) Exercise: show reduction works
4.3.1.2 Single-Source Shortest Paths via BFS

(A) **Special case:** All edge lengths are 1.
   (A) Run **BFS**\((s)\) to get shortest path distances from \(s\) to all other nodes.
   (B) **O**\((m + n)\) time algorithm.

(B) **Special case:** Suppose \(\ell(e)\) is an integer for all \(e\)?
   Can we use **BFS**? Reduce to unit edge-length problem by placing \(\ell(e) - 1\) dummy nodes on \(e\).

(C) Let \(L = \max_e \ell(e)\). New graph has \(O(mL)\) edges and \(O(mL + n)\) nodes. **BFS** takes \(O(mL + n)\) time. Not efficient if \(L\) is large.

4.3.1.3 Towards an algorithm

Why does **BFS** work?
**BFS**\((s)\) explores nodes in increasing distance from \(s\)

**Lemma 4.3.1.** Let \(G\) be a directed graph with non-negative edge lengths. Let \(\text{dist}(s, v)\) denote the shortest path length from \(s\) to \(v\). If \(s = v_0 \to v_1 \to v_2 \to \ldots \to v_k\) is a shortest path from \(s\) to \(v_k\) then for \(1 \leq i < k\):

(A) \(s = v_0 \to v_1 \to v_2 \to \ldots \to v_i\) is a shortest path from \(s\) to \(v_i\)
(B) \(\text{dist}(s, v_i) \leq \text{dist}(s, v_k)\).

**Proof:** Suppose not. Then for some \(i < k\) there is a path \(P'\) from \(s\) to \(v_i\) of length strictly less than that of \(s = v_0 \to v_1 \to \ldots \to v_i\). Then \(P'\) concatenated with \(v_i \to v_{i+1} \ldots \to v_k\) contains a strictly shorter path to \(v_k\) than \(s = v_0 \to v_1 \ldots \to v_k\). \(\blacksquare\)
4.3.1.4 A proof by picture

Shortest path from $v_0$ to $v_6$

A shorter path from $v_0$ to $v_6$. A contradiction.

4.3.1.5 A Basic Strategy

Explore vertices in increasing order of distance from $s$:

(For simplicity assume that nodes are at different distances from $s$ and that no edge has zero length)
Initialize for each node \( v \), \( \text{dist}(s,v) = \infty \)
Initialize \( S = \emptyset \),
for \( i = 1 \) to \( |V| \) do

(* Invariant: \( S \) contains the \( i - 1 \) closest nodes to \( s \) *)
Among nodes in \( V \setminus S \), find the node \( v \) that is the \( i \)th closest to \( s \)
Update \( \text{dist}(s,v) \)
\( S = S \cup \{v\} \)

How can we implement the step in the for loop?

4.3.1.6 Finding the \( i \)th closest node

(A) \( S \) contains the \( i - 1 \) closest nodes to \( s \)
(B) Want to find the \( i \)th closest node from \( V - S \).

What do we know about the \( i \)th closest node?

Claim 4.3.2. Let \( P \) be a shortest path from \( s \) to \( v \) where \( v \) is the \( i \)th closest node. Then, all intermediate nodes in \( P \) belong to \( S \).

Proof: If \( P \) had an intermediate node \( u \) not in \( S \) then \( u \) will be closer to \( s \) than \( v \). Implies \( v \) is not the \( i \)th closest node to \( s \) - recall that \( S \) already has the \( i - 1 \) closest nodes.

4.3.2 Finding the \( i \)th closest node repeatedly

4.3.2.1 An example

![Diagrams showing the process of finding the \( i \)th closest node repeatedly.](image-url)
4.3.2.2 Finding the $i$th closest node

Corollary 4.3.3. The $i$th closest node is adjacent to $S$.

4.3.2.3 Finding the $i$th closest node

(A) $S$ contains the $i - 1$ closest nodes to $s$
(B) Want to find the $i$th closest node from $V - S$.
(C) For each $u \in V \setminus S$ let $P(s, u, S)$ be a shortest path from $s$ to $u$ using only nodes in $S$ as intermediate vertices.
(D) Let $d'(s, u)$ be the length of $P(s, u, S)$
(E) Observations: for each $u \in V - S$,
   (A) $dist(s, u) \leq d'(s, u)$ since we are constraining the paths
   (B) $d'(s, u) = \min_{a \in S}(dist(s, a) + \ell(a, u))$ - Why?
(F) **Lemma 4.3.4.** If $v$ is the $i$th closest node to $s$, then $d'(s, v) = dist(s, v)$.

4.3.2.4 Finding the $i$th closest node

**Lemma 4.3.5.** Given:
(A) $S$: Set of $i - 1$ closest nodes to $s$.
(B) $d'(s, u) = \min_{x \in S}(dist(s, x) + \ell(x, u))$
If $v$ is an $i$th closest node to $s$, then $d'(s, v) = dist(s, v)$.

**Proof:** Let $v$ be the $i$th closest node to $s$. Then there is a shortest path $P$ from $s$ to $v$ that contains only nodes in $S$ as intermediate nodes (see previous claim). Therefore $d'(s, v) = dist(s, v)$.

4.3.2.5 Finding the $i$th closest node

**Lemma 4.3.6.** If $v$ is an $i$th closest node to $s$, then $d'(s, v) = dist(s, v)$.

**Corollary 4.3.7.** The $i$th closest node to $s$ is the node $v \in V - S$ such that $d'(s, v) = \min_{u \in V - S} d'(s, u)$.

**Proof:** For every node $u \in V - S$, $dist(s, u) \leq d'(s, u)$ and for the $i$th closest node $v$, $dist(s, v) = d'(s, v)$. Moreover, $dist(s, u) \geq dist(s, v)$ for each $u \in V - S$.
4.3.2.6 Candidate algorithm for shortest path

Initialize for each node \( v \): \( \text{dist}(s,v) = \infty \)
Initialize \( S = \emptyset \), \( d'(s,s) = 0 \)

for \( i = 1 \) to \(|V|\) do
    (* Invariant: \( S \) contains the \( i-1 \) closest nodes to \( s \) *)
    (* Invariant: \( d'(s,u) \) is shortest path distance from \( u \) to \( s \)
        using only \( S \) as intermediate nodes*)
    Let \( v \) be such that \( d'(s,v) = \min_{u \in V - S} d'(s,u) \)
    \( \text{dist}(s,v) = d'(s,v) \)
    \( S = S \cup \{v\} \)
    for each node \( u \) in \( V \setminus S \) do
        \( d'(s,u) = \min_{a \in S} \left( \text{dist}(s,a) + \ell(a,u) \right) \)

Correctness: By induction on \( i \) using previous lemmas.

Running time: \( O(n \cdot (n + m)) \) time.

(A) \( n \) outer iterations. In each iteration, \( d'(s,u) \) for each \( u \) by scanning all edges out of
nodes in \( S \); \( O(m + n) \) time/iteration.

4.3.2.7 Example

4.3.2.8 Improved Algorithm

(A) Main work is to compute the \( d'(s,u) \) values in each iteration
(B) \( d'(s,u) \) changes from iteration \( i \) to \( i + 1 \) only because of the node \( v \) that is added to \( S \)
in iteration \( i \).

Initialze for each node \( v \): \( \text{dist}(s,v) = d'(s,v) = \infty \)
Initialize \( S = \emptyset \), \( d'(s,s) = 0 \)

for \( i = 1 \) to \(|V|\) do
    // \( S \) contains the \( i-1 \) closest nodes to \( s \),
    // and the values of \( d'(s,u) \) are current
    \( v \) be node realizing \( d'(s,v) = \min_{u \in V - S} d'(s,u) \)
    \( \text{dist}(s,v) = d'(s,v) \)
    \( S = S \cup \{v\} \)
    Update \( d'(s,u) \) for each \( u \) in \( V - S \) as follows:
    \[ d'(s,u) = \min \left( d'(s,u), \text{dist}(s,v) + \ell(v,u) \right) \]

Running time: \( O(m + n^2) \) time.
(A) \(n\) outer iterations and in each iteration following steps
(B) updating \(d'(s, u)\) after \(v\) added takes \(O(\text{deg}(v))\) time so total work is \(O(m)\) since a node enters \(S\) only once
(C) Finding \(v\) from \(d'(s, u)\) values is \(O(n)\) time

4.3.2.9 Dijkstra’s Algorithm

(A) eliminate \(d'(s, u)\) and let \(\text{dist}(s, u)\) maintain it
(B) update \(\text{dist}\) values after adding \(v\) by scanning edges out of \(v\)

```
Initialize for each node \(v\), \(\text{dist}(s, v) = \infty\)
Initialize \(S = \emptyset\), \(\text{dist}(s, s) = 0\)
for \(i = 1\) to \(|V|\) do
    Let \(v\) be such that \(\text{dist}(s, v) = \min_{u \in V} \text{dist}(s, u)\)
    \(S = S \cup \{v\}\)
    for each \(u\) in \(\text{Adj}(v)\) do
        \(\text{dist}(s, u) = \min(\text{dist}(s, u), \text{dist}(s, v) + \ell(v, u))\)
```

Priority Queues to maintain \(\text{dist}\) values for faster running time

(A) Using heaps and standard priority queues: \(O((m + n) \log n)\)
(B) Using Fibonacci heaps: \(O(m + n \log n)\).

4.3.2.10 Example: Dijkstra algorithm in action

4.3.3 Priority Queues

4.3.3.1 Priority Queues

Data structure to store a set \(S\) of \(n\) elements where each element \(v \in S\) has an associated real/integer key \(k(v)\) such that the following operations:

(A) \text{makePQ}: create an empty queue.
(B) \text{findMin}: find the minimum key in \(S\).
(C) \text{extractMin}: Remove \(v \in S\) with smallest key and return it.
(D) \text{insert}(v, k(v)): Add new element \(v\) with key \(k(v)\) to \(S\).
(E) \text{delete}(v): Remove element \(v\) from \(S\).
(F) \text{decreaseKey}(v, k'(v)): decrease key of \(v\) from \(k(v)\) (current key) to \(k'(v)\) (new key).
    Assumption: \(k'(v) \leq k(v)\).
meld: merge two separate priority queues into one.
All operations can be performed in $O(\log n)$ time.
decreaseKey is implemented via delete and insert.

4.3.3.2 Dijkstra’s Algorithm using Priority Queues

```
Q ← makePQ()
insert(Q, (s, 0))
for each node $u \neq s$ do
    insert(Q, (u, ∞))
S ← ∅
for $i = 1$ to $|V|$ do
    $(v, \text{dist}(s, v)) = \text{extractMin}(Q)$
    $S = S \cup \{v\}$
    for each $u$ in Adj(v) do
        decreaseKey([[)]Q, (u, min(\text{dist}(s, u), \text{dist}(s, v) + \ell(v, u)))].
```

Priority Queue operations:
(A) $O(n)$ insert operations
(B) $O(n)$ extractMin operations
(C) $O(m)$ decreaseKey operations

4.3.3.3 Implementing Priority Queues via Heaps

Using Heaps Store elements in a heap based on the key value
(A) All operations can be done in $O(\log n)$ time
Dijkstra’s algorithm can be implemented in $O((n + m) \log n)$ time.

4.3.3.4 Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps
(A) extractMin, delete in $O(\log n)$ time.
(B) insert in $O(1)$ amortized time.
(C) decreaseKey in $O(1)$ amortized time: $\ell$ decreaseKey operations for $\ell \geq n$ take together $O(\ell)$ time
(D) Relaxed Heaps: decreaseKey in $O(1)$ worst case time but at the expense of meld (not necessary for Dijkstra’s algorithm)

(A) Dijkstra’s algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.
(B) Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)

4.3.3.5 Shortest Path Tree

Dijkstra’s algorithm finds the shortest path distances from $s$ to $V$.
Question: How do we find the paths themselves?
4.3.3.6 Shortest Path Tree

**Lemma 4.3.8.** The edge set \((u, \text{prev}(u))\) is the reverse of a shortest path tree rooted at \(s\). For each \(u\), the reverse of the path from \(u\) to \(s\) in the tree is a shortest path from \(s\) to \(u\).

**Proof:** [Proof Sketch.]

(A) The edge set \(\{(u, \text{prev}(u)) \mid u \in V\}\) induces a directed in-tree rooted at \(s\) (Why?)

(B) Use induction on \(|S|\) to argue that the tree is a shortest path tree for nodes in \(V\).

4.3.3.7 Shortest paths to \(s\)

Dijkstra’s algorithm gives shortest paths from \(s\) to all nodes in \(V\).

How do we find shortest paths from all of \(V\) to \(s\)?

(A) In undirected graphs shortest path from \(s\) to \(u\) is a shortest path from \(u\) to \(s\) so there is no need to distinguish.

(B) In directed graphs, use Dijkstra’s algorithm in \(G^{\text{rev}}\).