Dynamic Programming

Lecture 8

February 13, 2014

Part I

Longest Increasing Subsequence

Sequences

Definition

Sequence: an ordered list $a_1, a_2, \ldots, a_n$. **Length** of a sequence is number of elements in the list.

Definition

$a_{i_1}, \ldots, a_{i_k}$ is a **subsequence** of $a_1, \ldots, a_n$ if $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Definition

A sequence is **increasing** if $a_1 < a_2 < \ldots < a_n$. It is **non-decreasing** if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly **decreasing** and **non-increasing**.

Sequences

Example...

Example

1. Sequence: $6, 3, 5, 2, 7, 8, 1, 9$
2. Subsequence of above sequence: $5, 2, 1$
3. Increasing sequence: $3, 5, 9, 17, 54$
4. Decreasing sequence: $34, 21, 7, 5, 1$
5. Increasing <u>subsequence</u> of the first sequence: $2, 7, 9$.

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an **increasing subsequence** $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

Example

1. Sequence: 6, 3, 5, 2, 7, 8, 1
2. Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
3. Longest increasing subsequence: 3, 5, 7, 8

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Naïve Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array A

```
algLISNaive(A[1..n]):
    max = 0
    for each subsequence B of A do
        if B is increasing and |B| > max then
            max = |B|

    Output max
```

Running time: $O(n2^n)$.

2^n subsequences of a sequence of length n and $O(n)$ time to check if a given sequence is increasing.

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Recursive Approach: Take 1

LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

$\mathrm{LIS}(\mathbf{A[1..n]})$:

1. Case 1: Does not contain $\mathbf{A[n]}$ in which case $\mathrm{LIS}(\mathbf{A[1..n]}) = \mathrm{LIS}(\mathbf{A[1..(n-1)]})$
2. Case 2: contains $\mathbf{A[n]}$ in which case $\mathrm{LIS}(\mathbf{A[1..n]})$ is not so clear.

Observation

if $\mathbf{A[n]}$ is in the longest increasing subsequence then all the elements before it must be smaller.

Recursive Approach: Take 1

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if **A[n]** *is in the longest increasing subsequence then all the elements before it must be smaller.*

Recursive Approach: Take 1

```
algLIS(A[1..n]):
    if (n = 0) then return 0
    m = algLIS(A[1..(n − 1)])
    B is subsequence of A[1..(n − 1)] with
            only elements less than A[n]
    (* let h be size of B, h ≤ n − 1 *)
    m = max(m, 1 + algLIS(B[1..h]))
    Output m
```

Recursion for running time: $T(n) \leq 2T(n-1) + O(n)$.

Easy to see that $T(n)$ is $O(n2^n)$.

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How many different recursive calls does $\mathrm{algLIS}_1(A[1..n])$ <u>really</u> make?

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(A) $\Theta(n^2)$

(B) $\Theta(2^n)$

(C) $\Theta(n2^n)$

(D) $\Theta(2^{n^2})$

(E) $\Theta(n^n)$

Recursive Approach: Take 2

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Observation

For second case we want to find a subsequence in $\mathbf{A[1..(n-1)]}$ *that is restricted to numbers less than* $\mathbf{A[n]}$*. This suggests that a more general problem is* **LIS_smaller**$(\mathbf{A[1..n], x})$ *which gives the longest increasing subsequence in* $\mathbf{A}$ *where each number in the sequence is less than* $\mathbf{x}$*.*

Recursive Approach: Take 2

LIS_smaller(A[1..n], x) : length of longest increasing subsequence in **A[1..n]** with all numbers in subsequence less than **x**

```
LIS_smaller(A[1..n], x):
    if (n = 0) then return 0
    m = LIS_smaller(A[1..(n − 1)], x)
    if (A[n] < x) then
        m = max(m, 1 + LIS_smaller(A[1..(n − 1)], A[n]))
    Output m
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```
LIS(A[1..n]):
        return LIS_smaller(A[1..n], ∞)
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Question: Is there any advantage?

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Recursive Algorithm: Take 2

Observation

The number of different *subproblems generated by* $\mathbf{LIS\_smaller(A[1..n], x)}$ *is* $\mathbf{O(n^2)}$.

Memoization the recursive algorithm leads to an $\mathbf{O(n^2)}$ running time!

Question: What are the recursive subproblem generated by $\mathbf{LIS\_smaller(A[1..n], x)}$?

1. For $\mathbf{0 \leq i < n}$ $\mathbf{LIS\_smaller(A[1..i], y)}$ where $\mathbf{y}$ is either $\mathbf{x}$ or one of $\mathbf{A[i+1], \ldots, A[n]}$.

Observation

previous recursion also generates only $\mathbf{O(n^2)}$ *subproblems. Slightly harder to see.*

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Increasing/decreasing sequence

Given a sequence $a_1, a_2, \ldots, a_n$ of numbers, there is always either an increasing or decreasing subsequence of length (longer is better):

(A) $\Theta(1)$.

(B) $\Theta(\log \log n)$.

(C) $\Theta(\log n)$.

(D) $\Theta(\sqrt{n})$.

(E) $\Theta(n)$.

Recursive Algorithm: Take 3

Definition

LISEnding(A[1..n]): length of longest increasing sub-sequence that *ends* in **A[n]**.

Question: can we obtain a recursive expression?

$$\mathbf{LISEnding(A[1..n])} = \max_{i:A[i]<A[n]} \left(1 + \mathbf{LISEnding(A[1..i])}\right)$$

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Recursive Algorithm: Take 3

```
LIS_ending_alg(A[1..n]):
    if (n = 0) return 0
    m = 1
    for i = 1 to n − 1 do
        if (A[i] < A[n]) then
            m = max(m, 1 + LIS_ending_alg(A[1..i]))

    return m
```

```
LIS(A[1..n]):
        return max_{i=1}^{n} LIS_ending_alg(A[1 . . . i])
```

Question:

How many distinct subproblems generated by LIS_ending_alg(A[1..n])? n.

Recursive Algorithm: Take 3

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Iterative Algorithm via Memoization

Compute the values **LIS_ending_alg(A[1..i])** iteratively in a bottom up fashion.

```
LIS_ending_alg(A[1..n]):
    Array L[1..n]  (* L[i] = value of LIS_ending_alg(A[1..i]) *)
    for i = 1 to n do
        L[i] = 1
        for j = 1 to i − 1 do
            if (A[j] < A[i]) do
                L[i] = max(L[i], 1 + L[j])
    return L
```

```
LIS(A[1..n]):
        L = LIS_ending_alg(A[1..n])
        return the maximum value in L
```

Iterative Algorithm via Memoization

Simplifying:

```
LIS(A[1..n]):
    Array L[1..n]  (* L[i] stores the value LISEnding(A[1..i]) *)
    m = 0
    for i = 1 to n do
        L[i] = 1
        for j = 1 to i − 1 do
            if (A[j] < A[i]) do
                L[i] = max(L[i], 1 + L[j])
        m = max(m, L[i])
    return m
```

Correctness: Via induction following the recursion

Running time: $O(n^2)$, Space: $\Theta(n)$

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    m = 0
    for i = 1 to n do
        L[i] = 1
        for j = 1 to i - 1 do
            if (A[j] < A[i]) do
                L[i] = max(L[i], 1 + L[j])
        m = max(m, L[i])
    return m
```

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Example

Example

1. Sequence: 6, 3, 5, 2, 7, 8, 1
2. Longest increasing subsequence: 3, 5, 7, 8

$L[1] = 1$

$L[2] = 1$

$L[3] = \max\{1, 1+1\} = 2$

$L[4] = 1$

$L[5] = \max\{1, 1+1, 1+1, 1+2, 1+1\} = 3$

1. $\mathbf{L[i]}$ is value of longest increasing subsequence ending in $\mathbf{A[i]}$
2. Recursive algorithm computes $\mathbf{L[i]}$ from $\mathbf{L[1]}$ to $\mathbf{L[i-1]}$
3. Iterative algorithm builds up the values from $\mathbf{L[1]}$ to $\mathbf{L[n]}$

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Memoizing LIS_smaller

```
LIS(A[1..n]):
    A[n + 1] = ∞ (* add a sentinel at the end *)
    Array L[(n + 1), (n + 1)] (* two-dimensional array*)
        (* L[i, j] for j ≥ i stores the value LIS_smaller(A[1..i], A[j]) *
    for j = 1 to n + 1 do
        L[0, j] = 0
    for i = 1 to n + 1 do
        for j = i to n + 1 do
            L[i, j] = L[i − 1, j]
            if (A[i] < A[j]) then
                L[i, j] = max(L[i, j], 1 + L[i − 1, i])

    return L[n, (n + 1)]
```

Correctness: Via induction following the recursion (take 2)
Running time: $O(n^2)$, Space: $\Theta(n^2)$

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    A[n+1] = ∞ (* add a sentinel at the end *)
    Array L[(n+1),(n+1)] (* two-dimensional array*)
        (* L[i,j] for j ≥ i stores the value LIS_smaller(A[1..i],A[j]) *
    for j = 1 to n+1 do
        L[0,j] = 0
    for i = 1 to n+1 do
        for j = i to n+1 do
            L[i,j] = L[i-1,j]
            if (A[i] < A[j]) then
                L[i,j] = max(L[i,j], 1 + L[i-1,i])

    return L[n,(n+1)]
```

Correctness: Via induction following the recursion (take 2)
Running time: $O(n^2)$, Space: $\Theta(n^2)$

Longest increasing subsequence

Another way to get quadratic time algorithm

1. $G = (\{s, 1, \ldots, n\}, \{\})$: directed graph.
   1. $\forall i, j$: If $i < j$ and $A[i] < A[j]$ then
      add the edge $i \to j$ to G.
   2. $\forall i$: Add $s \to i$.
2. The graph G is a DAG. LIS corresponds to longest path in G starting at s.
3. We know how to compute this in $O(|V(G)| + |E(G)|) = O(n^2)$.

Comment: One can compute LIS in $O(n \log n)$ time with a bit more work.

Dynamic Programming

1. Find a "smart" recursion for the problem in which the number of distinct subproblems is small; polynomial in the original problem size.
2. Estimate the number of subproblems, the time to evaluate each subproblem and the space needed to store the value. This gives an upper bound on the total running time if we use automatic memoization.
3. Eliminate recursion and find an iterative algorithm to compute the problems bottom up by storing the intermediate values in an appropriate data structure; need to find the right way or order the subproblem evaluation. This leads to an explicit algorithm.
4. Optimize the resulting algorithm further

Part II

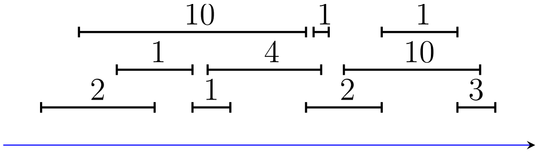
Weighted Interval Scheduling

Weighted Interval Scheduling

Input A set of jobs with start times, finish times and *weights* (or profits).

Goal Schedule jobs so that total weight of jobs is maximized.

1. Two jobs with overlapping intervals cannot both be scheduled!



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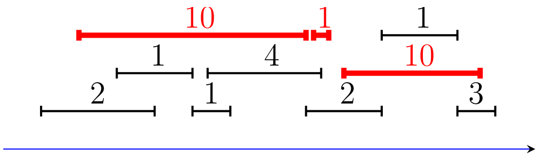


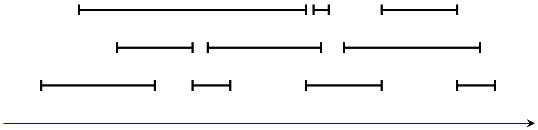
Interval Scheduling

Greedy Solution

Input A set of jobs with start and finish times to be scheduled on a resource; special case where all jobs have weight 1.

Goal Schedule as many jobs as possible.

1. Greedy strategy of considering jobs according to finish times produces optimal schedule (to be seen later).



Interval Scheduling

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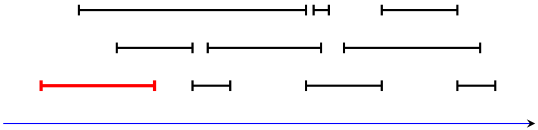

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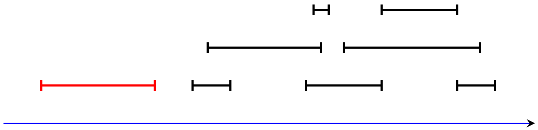

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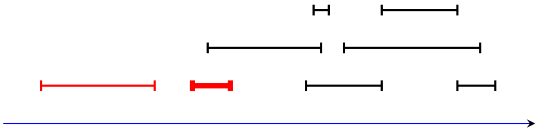

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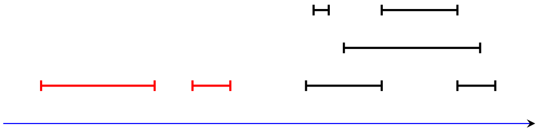

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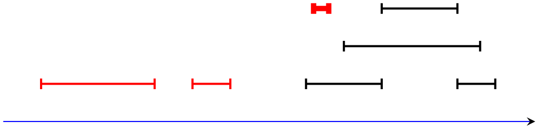

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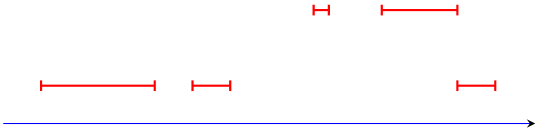

Greedy Strategies

1. Earliest finish time first
2. Largest weight/profit first
3. Largest weight to length ratio first
4. Shortest length first
5. ...

None of the above strategies lead to an optimum solution.

Moral: Greedy strategies often don't work!

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Reduction to...

Max Weight Independent Set Problem

1. Given weighted interval scheduling instance I create an instance of max weight independent set on a graph $G(I)$ as follows.
    1. For each interval i create a vertex v_i with weight w_i.
    2. Add an edge between v_i and v_j if i and j overlap.
2. Claim: max weight independent set in $G(I)$ has weight equal to max weight set of intervals in I that do not overlap

Reduction to...

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Reduction to...

Max Weight Independent Set Problem

1. There is a reduction from **Weighted Interval Scheduling** to **Independent Set**.
2. Can use structure of original problem for efficient algorithm?
3. **Independent Set** in general is **NP-Complete**.

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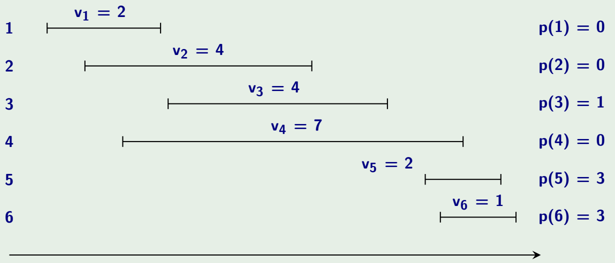
3. **Independent Set** in general is **NP-Complete**.

Conventions

Definition

1. Let the requests be sorted according to finish time, i.e., $i < j$ implies $f_i \leq f_j$
2. Define $p(j)$ to be the largest i (less than j) such that job i and job j are not in conflict

Example



Towards a Recursive Solution

Observation

Consider an optimal schedule $\mathcal{O}$

Case $\mathbf{n} \in \mathcal{O}$: *None of the jobs between* $\mathbf{n}$ *and* $\mathbf{p(n)}$ *can be scheduled. Moreover* $\mathcal{O}$ *must contain an optimal schedule for the first* $\mathbf{p(n)}$ *jobs.*

Case $\mathbf{n} \notin \mathcal{O}$: $\mathcal{O}$ *is an optimal schedule for the first* $\mathbf{n-1}$ *jobs.*

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A Recursive Algorithm

Let O_i be value of an optimal schedule for the first i jobs.

```
Schedule(n):
    if n = 0 then return 0
    if n = 1 then return w(v_1)
    O_{p(n)} ← Schedule(p(n))
    O_{n−1} ← Schedule(n − 1)
    if (O_{p(n)} + w(v_n) < O_{n−1}) then
        O_n = O_{n−1}
    else
        O_n = O_{p(n)} + w(v_n)
    return O_n
```

Time Analysis

Running time is $T(n) = T(p(n)) + T(n-1) + O(1)$ which is ...

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The solution to the following recurrence is?

$\mathbf{T(n) = T(n-2) + T(n-17) + 65}$

(A) $2^{\Theta(n)}$.

(B) $\Theta(n)$.

(C) 65.

(D) $\Theta(F_n)$, where F_n is the nth Fibonacci number..

(E) $\Theta(0)$.

Bad Example

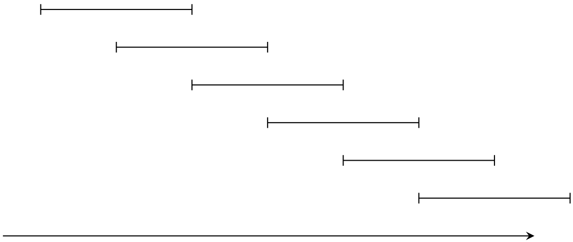

Figure : Bad instance for recursive algorithm

Running time on this instance is

$$\mathbf{T(n) = T(n-1) + T(n-2) + O(1) = \Theta(\phi^n)}$$

where $\phi \approx 1.618$ is the golden ratio.

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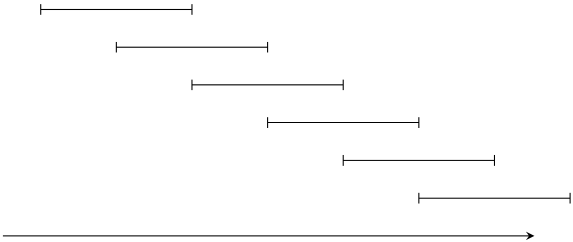

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Analysis of the Problem

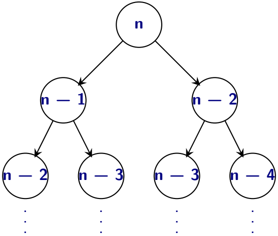

Figure : Label of node indicates size of sub-problem. Tree of sub-problems grows very quickly

Memo(r)ization

Observation

1. *Number of different sub-problems in recursive algorithm is* $\mathbf{O(n)}$*; they are* $\mathbf{O_1, O_2, \ldots, O_{n-1}}$
2. *Exponential time is due to recomputation of solutions to sub-problems*

Solution

Store optimal solution to different sub-problems, and perform recursive call only if not already computed.

Recursive Solution with Memoization

```
schdIMem(j)
    if j = 0 then return 0
    if M[j] is defined then (* sub-problem already solved *)
        return M[j]
    if M[j] is not defined then
        M[j] = max( w(v_j) + schdIMem(p(j)),   schdIMem(j − 1) )
        return M[j]
```

Time Analysis

- Each invocation, $O(1)$ time plus: either return a computed value, or generate 2 recursive calls and fill one $M[\cdot]$
- Initially no entry of $M[]$ is filled; at the end all entries of $M[]$ are filled
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Automatic Memoization

Fact

Many functional languages (like LISP) automatically do memoization for recursive function calls!

Back to Weighted Interval Scheduling

Iterative Solution

$$\begin{array}{l} M[0] = 0 \\ \textbf{for } i = 1 \texttt{ to } n \textbf{ do} \\ \quad M[i] = \max\Big(w(v_i) + M[p(i)], M[i-1]\Big) \end{array}$$

M: table of subproblems

1. Implicitly dynamic programming fills the values of M.
2. Recursion determines order in which table is filled up.
3. Think of decomposing problem first (recursion) and then worry about setting up table — this comes naturally from recursion.

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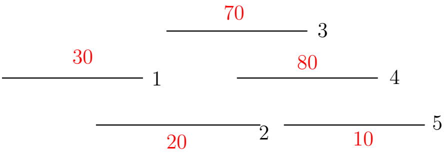
Example



$p(5) = 2,\ p(4) = 1,\ p(3) = 1,\ p(2) = 0,\ p(1) = 0$

$M[0] = 0 \quad D[0] = 0$

$M[1] = 30 \quad D[1] = 1$

$M[2] = \max\{M[1], 20 + 0\} = 30 \quad D[2] = 0$

$M[3] = \max\{M[2], 70 + M[1]\} = 100 \quad D[3] = 1$

$M[4] = \max\{M[3], 80 + M[1]\} = 110 \quad D[4] = 1$

$M[5] = \max\{M[4], 10 + M[2]\} = 110 \quad D[5] = 0$

Computing Solutions + First Attempt

1. Memoization + Recursion/Iteration allows one to compute the optimal value. What about the actual schedule?

```
M[0] = 0
S[0] is empty schedule
for i = 1 to n do
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Computing Implicit Solutions

Observation

Solution can be obtained from **M[]** *in* $O(n)$ *time, without any additional information*

```
findSolution( j )
    if (j = 0) then return empty schedule
    if (v_j + M[p(j)] > M[j − 1]) then
        return findSolution(p(j)) ∪ {j}
    else
        return findSolution(j − 1)
```

Makes $O(n)$ *recursive calls, so* **findSolution** *runs in* $O(n)$ *time.*

Computing Implicit Solutions

A generic strategy for computing solutions in dynamic programming:

1. Keep track of the *decision* in computing the optimum value of a sub-problem. decision space depends on recursion
2. Once the optimum values are computed, go back and use the decision values to compute an optimum solution.

Question: What is the decision in computing **M[i]**?

A: Whether to include **i** or not.

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    if (v_i + M[p(i)] > M[i − 1])then
        Decision[i] = 1 (* 1: i included in solution M[i] *)
    else
        Decision[i] = 0 (* 0: i not included in solution M[i] *)

S = ∅, i = n
while (i > 0) do
    if (Decision[i] = 1) then
        S = S ∪ {i}
        i = p(i)
    else
        i = i − 1
return S
```

Running time with memoization?

If we memoize the following function, what would be the running time of the resulting function, if we call **Confused**$(\mathbf{n}, \mathbf{n})$?

Confused(x, y)
    if $x > y$ or $x < 0$ **then if** $x = 0$ **then return** $2y$
    $\alpha =$ **Confused**$(x-1, y)$, $\beta =$ **Confused**$(x-1, y-1)$,
    $\gamma =$ **Confused**$(x-1, y-1)$, $\delta =$ **Confused**$(x-1, y-17)$,
    $\mu =$ **Confused**$(x-32, y-17)$,
    return $1 + \max(\alpha, \beta, \gamma, \delta, \mu)$

(A) $\Theta(n)$
(B) $\Theta(n^2)$
(C) $\Theta(n^3)$
(D) $\Theta(n^4)$
(E) $\Theta(n^5)$

Notes

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