Breadth First Search, Dijkstra's Algorithm for Shortest Paths

Lecture 3

January 28, 2014

Part I

Breadth First Search

Breadth First Search (BFS)

Overview

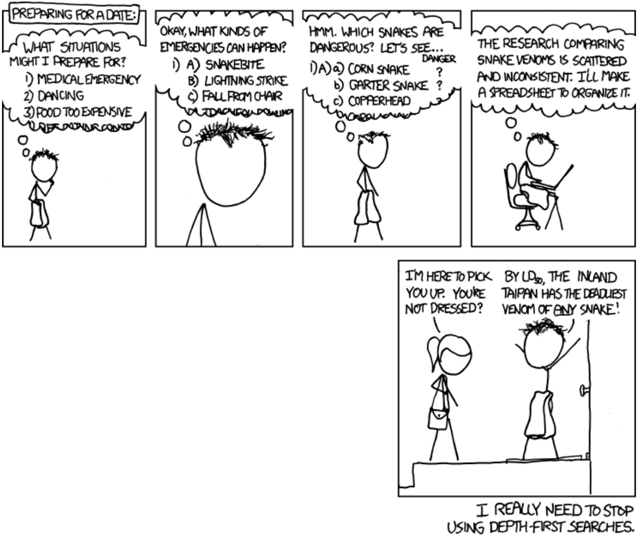
(A) **BFS** is obtained from **BasicSearch** by processing edges using a data structure called a **queue**.

(B) It processes the vertices in the graph in the order of their shortest distance from the vertex **s** (the start vertex).

As such...

1. **DFS** good for exploring graph structure
2. **BFS** good for exploring *distances*

xkcd take on DFS



Queue Data Structure

Queues

A **queue** is a list of elements which supports the operations:

1. **enqueue**: Adds an element to the end of the list
2. **dequeue**: Removes an element from the front of the list

Elements are extracted in **first-in first-out (FIFO)** order, i.e., elements are picked in the order in which they were inserted.

BFS Algorithm

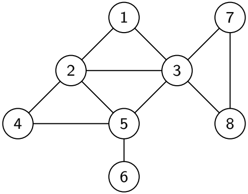
Given (undirected or directed) graph $G = (V, E)$ and node $s \in V$

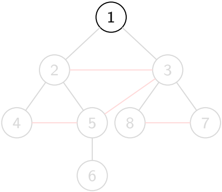
```
BFS(s)
    Mark all vertices as unvisited
    Initialize search tree T to be empty
    Mark vertex s as visited
    set Q to be the empty queue
    enq(s)
    while Q is nonempty do
        u = deq(Q)
        for each vertex v ∈ Adj(u)
            if v is not visited then
                add edge (u,v) to T
                Mark v as visited and enq(v)
```

Proposition

$BFS(s)$ *runs in* $O(n + m)$ *time.*

BFS: An Example in Undirected Graphs





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BFS tree is the set of black edges.

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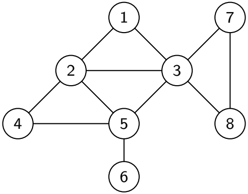

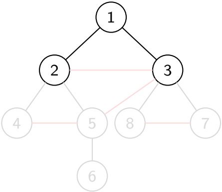

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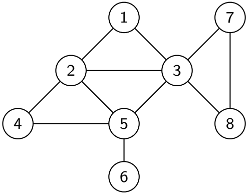

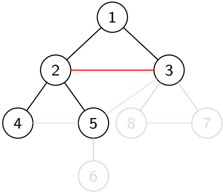

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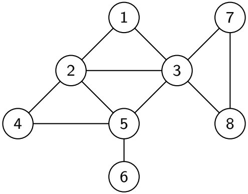

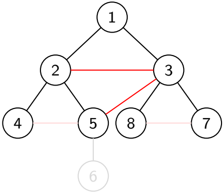

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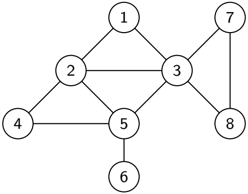

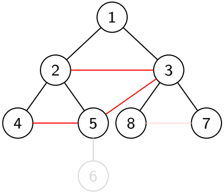

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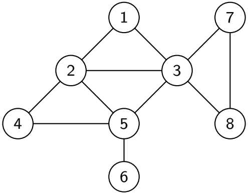

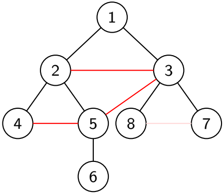

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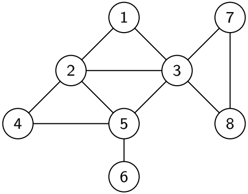

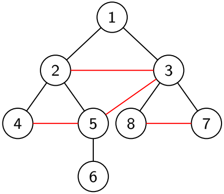

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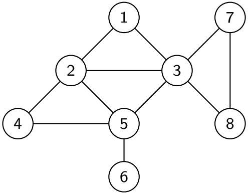

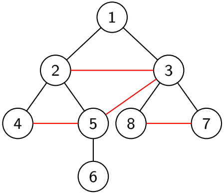

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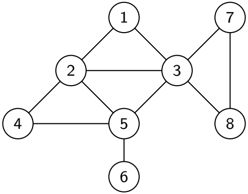

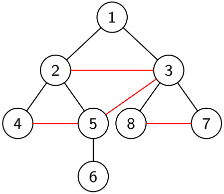

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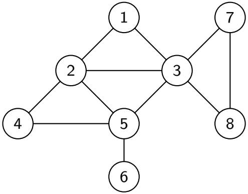

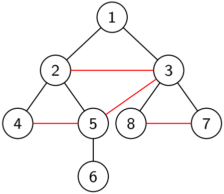

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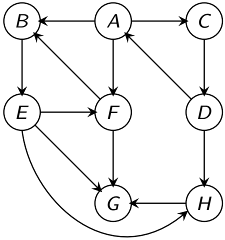
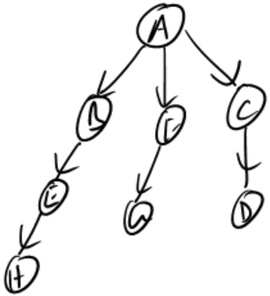
BFS: An Example in Directed Graphs





BFS with Distance

```
BFS(s)
    Mark all vertices as unvisited; for each v set dist(v) = ∞
    Initialize search tree T to be empty
    Mark vertex s as visited and set dist(s) = 0
    set Q to be the empty queue
    enq(s)
    while Q is nonempty do
        u = deq(Q)
        for each vertex v ∈ Adj(u) do
            if v is not visited do
                add edge (u, v) to T
                Mark v as visited, enq(v)
                and set dist(v) = dist(u) + 1
```

Properties of BFS: Undirected Graphs

Proposition

The following properties hold upon termination of **BFS**$(\mathbf{s})$

(A) *The search tree contains exactly the set of vertices in the connected component of* $\mathbf{s}$.

(B) *If* $\mathrm{dist}(\mathbf{u}) < \mathrm{dist}(\mathbf{v})$ *then* $\mathbf{u}$ *is visited before* $\mathbf{v}$.

(C) *For every vertex* $\mathbf{u}$, $\mathrm{dist}(\mathbf{u})$ *is the length of a shortest path (in terms of of edges) from* $\mathbf{s}$ *to* $\mathbf{u}$.

(D) *If* $\mathbf{u}, \mathbf{v}$ *are in connected component of* $\mathbf{s}$ *and* $\mathbf{e} = \{\mathbf{u}, \mathbf{v}\}$ *is an edge of* $\mathbf{G}$, *then* $|\mathrm{dist}(\mathbf{u}) - \mathrm{dist}(\mathbf{v})| \leq 1$.

Proof.

Exercise. □

Properties of BFS: Directed Graphs

Proposition

The following properties hold upon termination of **BFS**$(\mathbf{s})$*:*

(A) *The search tree contains exactly the set of vertices reachable from* $\mathbf{s}$

(B) *If* $\mathrm{dist}(\mathbf{u}) < \mathrm{dist}(\mathbf{v})$ *then* $\mathbf{u}$ *is visited before* $\mathbf{v}$

(C) *For every vertex* $\mathbf{u}$, $\mathrm{dist}(\mathbf{u})$ *is indeed the length of shortest path from* $\mathbf{s}$ *to* $\mathbf{u}$

(D) *If* $\mathbf{u}$ *is reachable from* $\mathbf{s}$ *and* $\mathbf{e} = (\mathbf{u}, \mathbf{v})$ *is an edge of* $\mathbf{G}$*, then* $\mathrm{dist}(\mathbf{v}) - \mathrm{dist}(\mathbf{u}) \leq 1$. *Not necessarily the case that* $\mathrm{dist}(\mathbf{u}) - \mathrm{dist}(\mathbf{v}) \leq 1$.

Proof.

Exercise. □

BFS with Layers

```
BFSLayers(s):
    Mark all vertices as unvisited and initialize T to be empty
    Mark s as visited and set L_0 = {s}
    i = 0
    while L_i is not empty do
            initialize L_{i+1} to be an empty list
            for each u in L_i do
                for each edge (u, v) ∈ Adj(u) do
                if v is not visited
                        mark v as visited
                        add (u, v) to tree T
                        add v to L_{i+1}
            i = i + 1
```

Running time: $O(n + m)$

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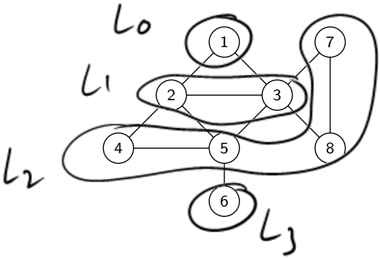
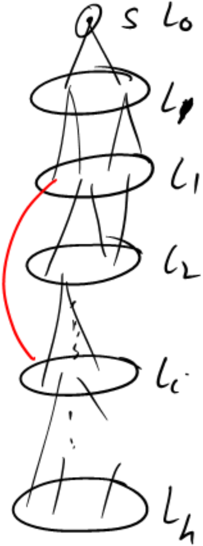
Example





BFS with Layers: Properties

Proposition

The following properties hold on termination of **BFSLayers**$(\mathbf{s})$.

1. **BFSLayers**$(\mathbf{s})$ *outputs a* **BFS** *tree*
2. $\mathbf{L_i}$ *is the set of vertices at distance exactly* $\mathbf{i}$ *from* $\mathbf{s}$
3. *If* $\mathbf{G}$ *is undirected, each edge* $\mathbf{e} = \{\mathbf{u}, \mathbf{v}\}$ *is one of three types:*
    1. **tree** *edge between two* consecutive *layers*
    2. *non-tree* **forward/backward** *edge between two consecutive layers*
    3. *non-tree* **cross-edge** *with both* $\mathbf{u}, \mathbf{v}$ *in same layer*
    4. $\Longrightarrow$ *Every edge in the graph is either between two vertices that are either (i) in the same layer, or (ii) in two consecutive layers.*

BFS with Layers: Properties

For directed graphs

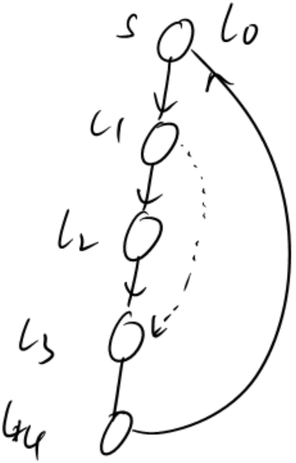
Proposition

The following properties hold on termination of **BFSLayers**(s), *if* G *is directed.*

For each edge $e = (u, v)$ *is one of four types:*

1. *a* **tree** *edge between consecutive layers,* $u \in L_i, v \in L_{i+1}$ *for some* $i \geq 0$
2. *a non-tree* **forward** *edge between consecutive layers*
3. *a non-tree* **backward** *edge*
4. *a* **cross-edge** *with both* u, v *in same layer*

Example



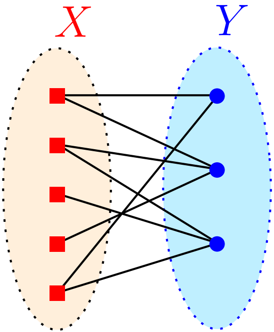
Part II

Bipartite Graphs and an application of BFS

Bipartite Graphs

Definition (Bipartite Graph)

Undirected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ is a **bipartite graph** if $\mathbf{V}$ can be partitioned into $\mathbf{X}$ and $\mathbf{Y}$ s.t. all edges in $\mathbf{E}$ are between $\mathbf{X}$ and $\mathbf{Y}$.



Bipartite Graph Characterization

Question

When is a graph bipartite?

Proposition

Every tree is a bipartite graph.

Proof.

Root tree T at some node r. Let L_i be all nodes at level i, that is, L_i is all nodes at distance i from root r. Now define X to be all nodes at even levels and Y to be all nodes at odd level. Only edges in T are between levels. □

Proposition

An odd length cycle is not bipartite.

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Odd Cycles are not Bipartite

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An odd length cycle is not bipartite.

Proof.

Let $\mathbf{C} = \mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_{2k+1}}, \mathbf{u_1}$ be an odd cycle. Suppose $\mathbf{C}$ is a bipartite graph and let $\mathbf{X}, \mathbf{Y}$ be the partition. Without loss of generality $\mathbf{u_1} \in \mathbf{X}$. Implies $\mathbf{u_2} \in \mathbf{Y}$. Implies $\mathbf{u_3} \in \mathbf{X}$. Inductively, $\mathbf{u_i} \in \mathbf{X}$ if $\mathbf{i}$ is odd $\mathbf{u_i} \in \mathbf{Y}$ if $\mathbf{i}$ is even. But $\{\mathbf{u_1}, \mathbf{u_{2k+1}}\}$ is an edge and both belong to $\mathbf{X}$! □

Subgraphs

Definition

Given a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ a **subgraph** of $\mathbf{G}$ is another graph $\mathbf{H} = (\mathbf{V}', \mathbf{E}')$ where $\mathbf{V}' \subseteq \mathbf{V}$ and $\mathbf{E}' \subseteq \mathbf{E}$.

Proposition

If $\mathbf{G}$ is bipartite then any subgraph $\mathbf{H}$ of $\mathbf{G}$ is also bipartite.

Proposition

A graph $\mathbf{G}$ is not bipartite if $\mathbf{G}$ has an odd cycle $\mathbf{C}$ as a subgraph.

Proof.

If $\mathbf{G}$ is bipartite then since $\mathbf{C}$ is a subgraph, $\mathbf{C}$ is also bipartite (by above proposition). However, $\mathbf{C}$ is not bipartite! □

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Bipartite Graph Characterization

Theorem

A graph $\mathbf{G}$ is bipartite if and only if it has no odd length cycle as subgraph.

Proof.

Only If: $\mathbf{G}$ has an odd cycle implies $\mathbf{G}$ is not bipartite.

If: $\mathbf{G}$ has no odd length cycle. Assume without loss of generality that $\mathbf{G}$ is connected.

1. Pick $\mathbf{u}$ arbitrarily and do **BFS**($\mathbf{u}$)
2. $\mathbf{X} = \cup_{\mathbf{i} \text{ is even}} \mathbf{L_i}$ and $\mathbf{Y} = \cup_{\mathbf{i} \text{ is odd}} \mathbf{L_i}$
3. **Claim:** $\mathbf{X}$ and $\mathbf{Y}$ is a valid partition if $\mathbf{G}$ has no odd length cycle.

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□

Proof of Claim

Claim

In **BFS**$(\mathbf{u})$ *if* $\mathbf{a}, \mathbf{b} \in \mathbf{L_i}$ *and* $(\mathbf{a}, \mathbf{b})$ *is an edge then there is an odd length cycle containing* $(\mathbf{a}, \mathbf{b})$*.*

Proof.

Let $\mathbf{v}$ be least common ancestor of $\mathbf{a}, \mathbf{b}$ in **BFS** tree $\mathbf{T}$.
$\mathbf{v}$ is in some level $\mathbf{j} < \mathbf{i}$ (could be $\mathbf{u}$ itself).
Path from $\mathbf{v} \rightsquigarrow \mathbf{a}$ in $\mathbf{T}$ is of length $\mathbf{i} - \mathbf{j}$.
Path from $\mathbf{v} \rightsquigarrow \mathbf{b}$ in $\mathbf{T}$ is of length $\mathbf{i} - \mathbf{j}$.
These two paths plus $(\mathbf{a}, \mathbf{b})$ forms an odd cycle of length $\mathbf{2(i - j) + 1}$. □

Proof of Claim

Claim

In **BFS**$(\mathbf{u})$ *if* $\mathbf{a}, \mathbf{b} \in \mathbf{L_i}$ *and* $(\mathbf{a}, \mathbf{b})$ *is an edge then there is an odd length cycle containing* $(\mathbf{a}, \mathbf{b})$.

Proof.

Let $\mathbf{v}$ be least common ancestor of $\mathbf{a}, \mathbf{b}$ in **BFS** tree $\mathbf{T}$.
$\mathbf{v}$ is in some level $\mathbf{j} < \mathbf{i}$ (could be $\mathbf{u}$ itself).
Path from $\mathbf{v} \rightsquigarrow \mathbf{a}$ in $\mathbf{T}$ is of length $\mathbf{i} - \mathbf{j}$.
Path from $\mathbf{v} \rightsquigarrow \mathbf{b}$ in $\mathbf{T}$ is of length $\mathbf{i} - \mathbf{j}$.
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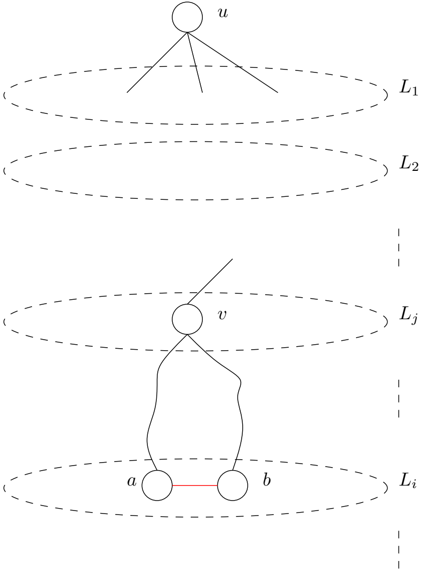
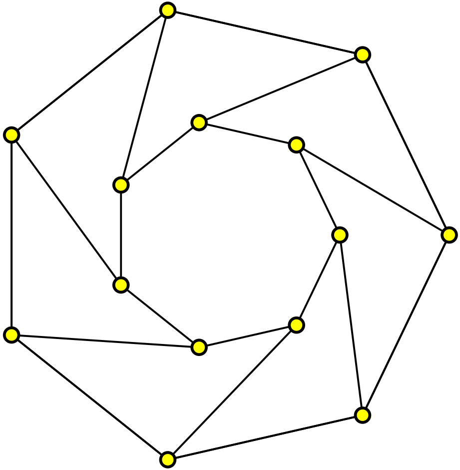
Proof of Claim: Figure



Is this graph bipartite?



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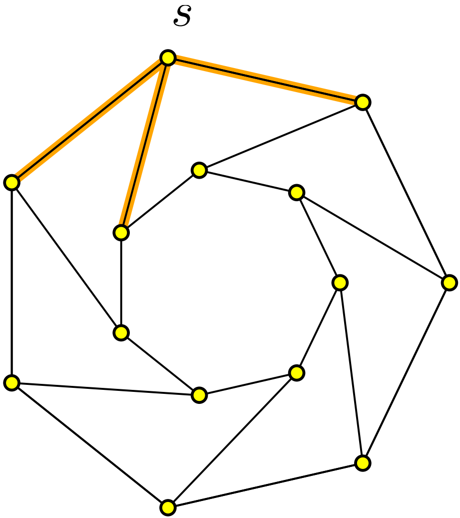

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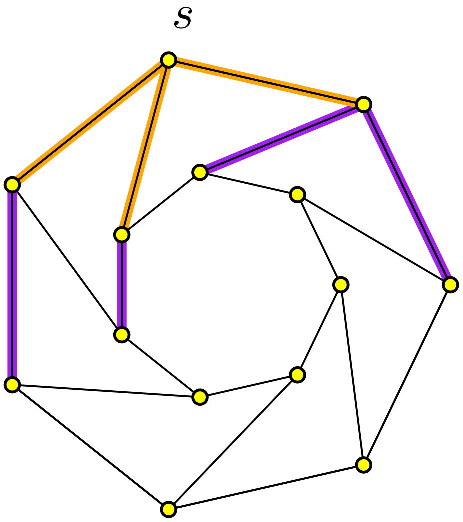

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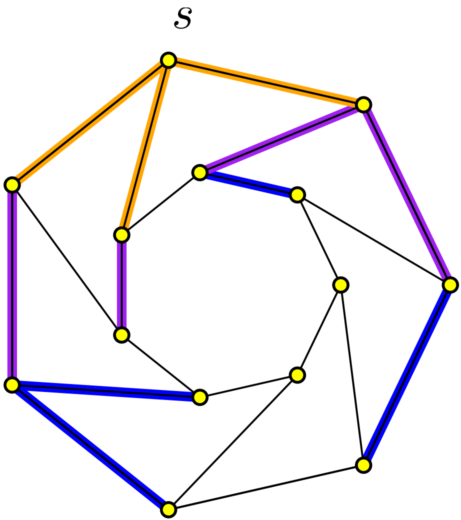

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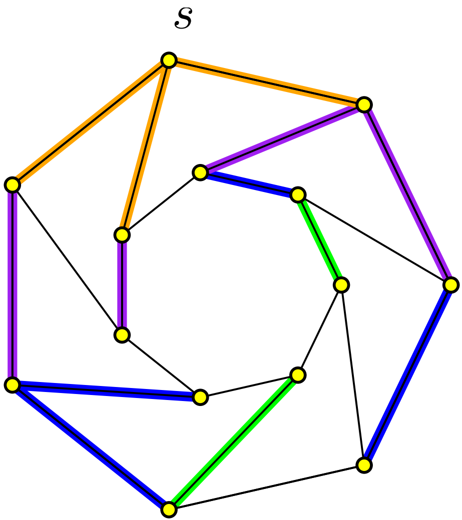

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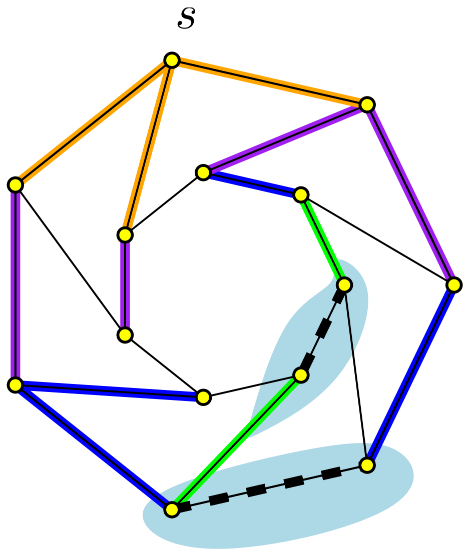

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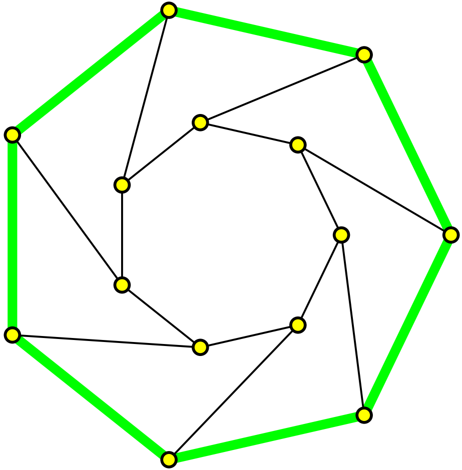

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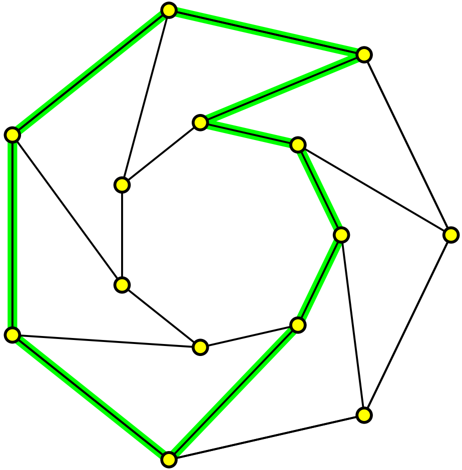

Another tidbit

Corollary

There is an $\mathbf{O(n + m)}$ *time algorithm to check if* $\mathbf{G}$ *is bipartite and output an odd cycle if it is not.*

Question: Can you come up with an efficient algorithm to check whether a given graph $\mathbf{G}$ has an even length cycle and find one if it has? What is the running time of your algorithm?

Part III

Shortest Paths and Dijkstra's Algorithm

Shortest Path Problems

Shortest Path Problems

Input A (undirected or directed) graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with edge lengths (or costs). For edge $\mathbf{e} = (\mathbf{u}, \mathbf{v})$, $\ell(\mathbf{e}) = \ell(\mathbf{u}, \mathbf{v})$ is its length.

1. Given nodes $\mathbf{s}, \mathbf{t}$ find shortest path from $\mathbf{s}$ to $\mathbf{t}$.
2. Given node $\mathbf{s}$ find shortest path from $\mathbf{s}$ to all other nodes.
3. Find shortest paths for all pairs of nodes.

Many applications!

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Single-Source Shortest Paths:

Non-Negative Edge Lengths

Single-Source Shortest Path Problems

1. Input: A (undirected or directed) graph $G = (V, E)$ with non-negative edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.
2. Given nodes s, t find shortest path from s to t.
3. Given node s find shortest path from s to all other nodes.

1. Restrict attention to directed graphs
2. Undirected graph problem can be reduced to directed graph problem - how?
   1. Given undirected graph G, create a new directed graph G' by replacing each edge $\{u, v\}$ in G by (u, v) and (v, u) in G'.
   2. set $\ell(u, v) = \ell(v, u) = \ell(\{u, v\})$
   3. Exercise: show reduction works. Relies on non-negativity!

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Single-Source Shortest Paths via BFS

Special case: All edge lengths are 1.

1. Run **BFS**(s) to get shortest path distances from s to all other nodes.
2. $O(m + n)$ time algorithm.

Special case: Suppose $\ell(e)$ is an integer for all e?
Can we use **BFS**? Reduce to unit edge-length problem by placing $\ell(e) - 1$ dummy nodes on e

Let $L = \max_e \ell(e)$. New graph has $O(mL)$ edges and $O(mL + n)$ nodes. **BFS** takes $O(mL + n)$ time. Not efficient if L is large.

Single-Source Shortest Paths via BFS

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Towards an algorithm

Why does **BFS** work?

BFS(s) explores nodes in increasing distance from s

Lemma

Let G be a directed graph with non-negative edge lengths. Let $\mathrm{dist}(s, v)$ denote the shortest path length from s to v. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from s to v_k then for $1 \leq i < k$:

1. *$s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from s to v_i*
2. *$\mathrm{dist}(s, v_i) \leq \mathrm{dist}(s, v_k)$.*

Proof.

Suppose not. Then for some $i < k$ there is a path P' from s to v_i of length strictly less than that of $s = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_i$. Then P' concatenated with $v_i \rightarrow v_{i+1} \ldots \rightarrow v_k$ contains a strictly shorter

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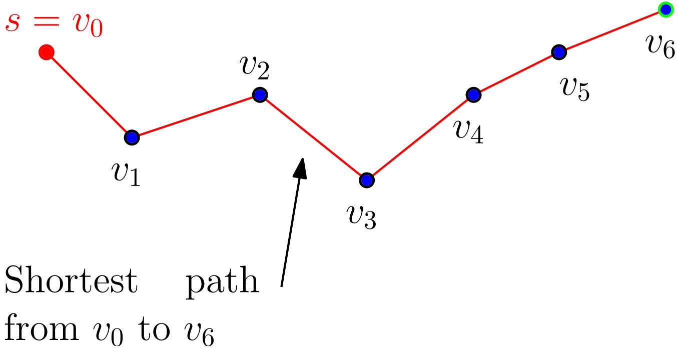
A proof by picture



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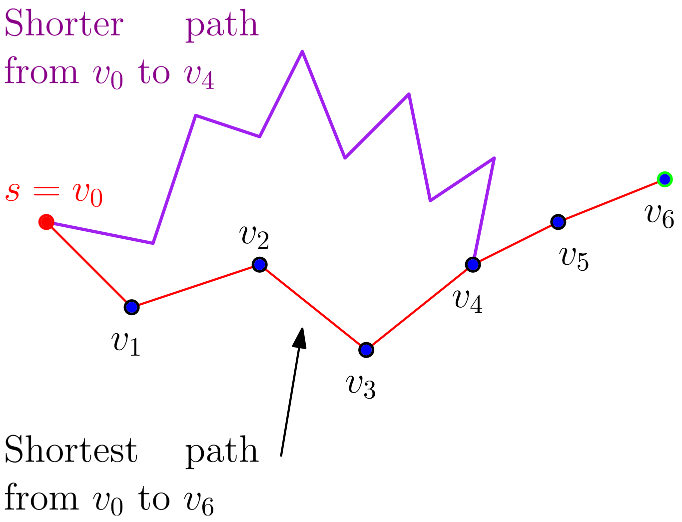

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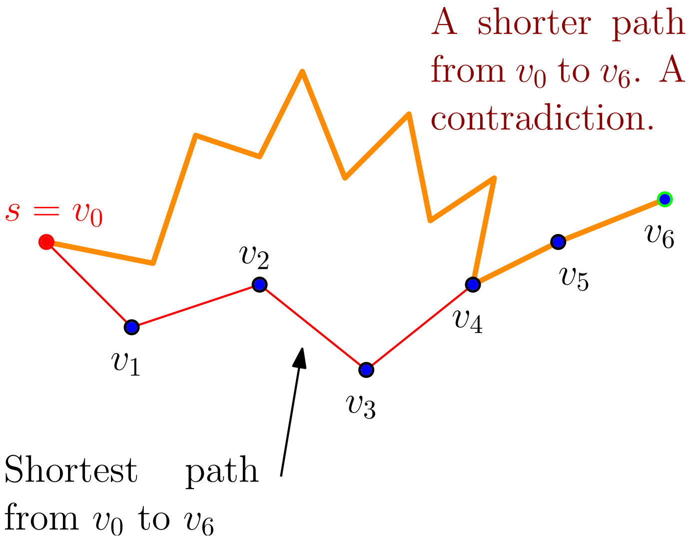

A Basic Strategy

Explore vertices in increasing order of distance from s:
(For simplicity assume that nodes are at different distances from s and that no edge has zero length)

```
Initialize for each node v, dist(s, v) = ∞
Initialize S = ∅,
for i = 1 to |V| do
    (* Invariant:  S contains the i − 1 closest nodes to s *)
    Among nodes in V \ S, find the node v that is the
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    Update dist(s, v)
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How can we implement the step in the for loop?

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Finding the ith closest node

1. $\mathbf{S}$ contains the $\mathbf{i-1}$ closest nodes to $\mathbf{s}$
2. Want to find the $\mathbf{i}$th closest node from $\mathbf{V-S}$.

What do we know about the $\mathbf{i}$th closest node?

Claim

Let $\mathbf{P}$ be a shortest path from $\mathbf{s}$ to $\mathbf{v}$ where $\mathbf{v}$ is the $\mathbf{i}$th closest node. Then, all intermediate nodes in $\mathbf{P}$ belong to $\mathbf{S}$.

Proof.

If $\mathbf{P}$ had an intermediate node $\mathbf{u}$ not in $\mathbf{S}$ then $\mathbf{u}$ will be closer to $\mathbf{s}$ than $\mathbf{v}$. Implies $\mathbf{v}$ is not the $\mathbf{i}$th closest node to $\mathbf{s}$ - recall that $\mathbf{S}$ already has the $\mathbf{i-1}$ closest nodes. □

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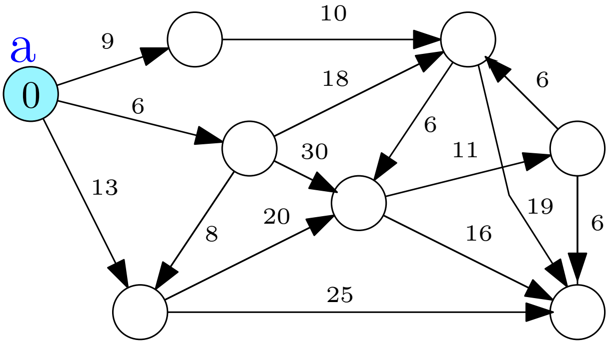
Finding the ith closest node repeatedly

An example



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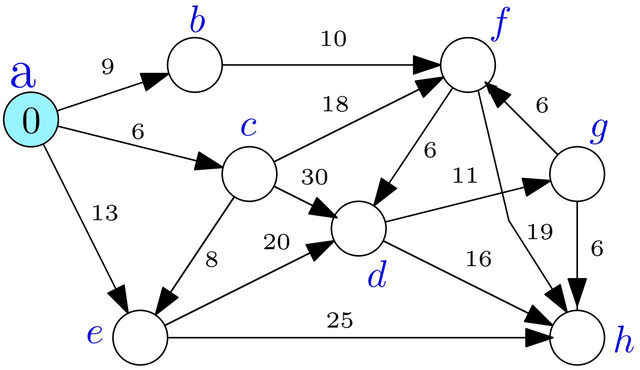

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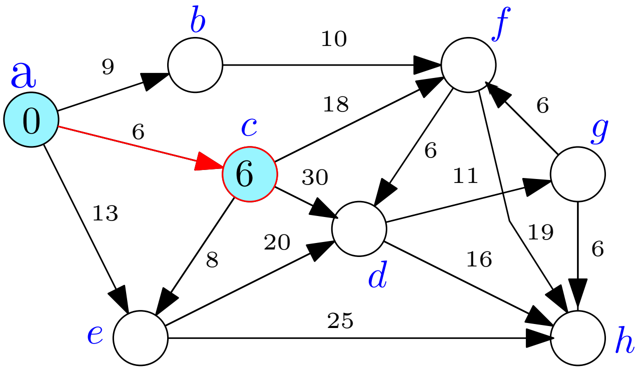

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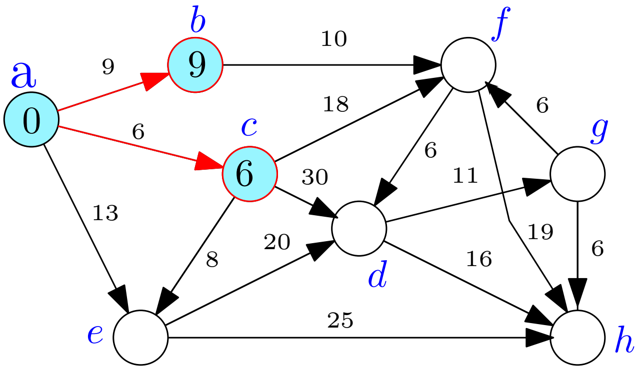

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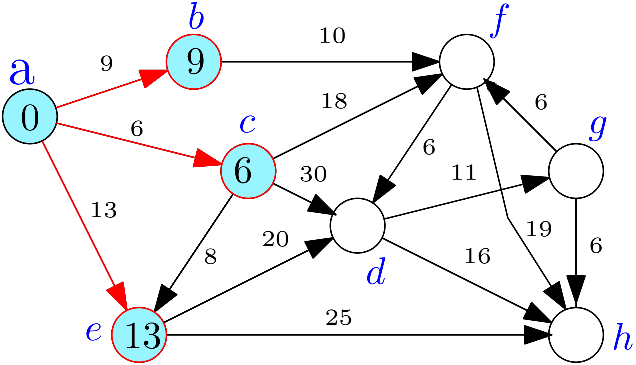

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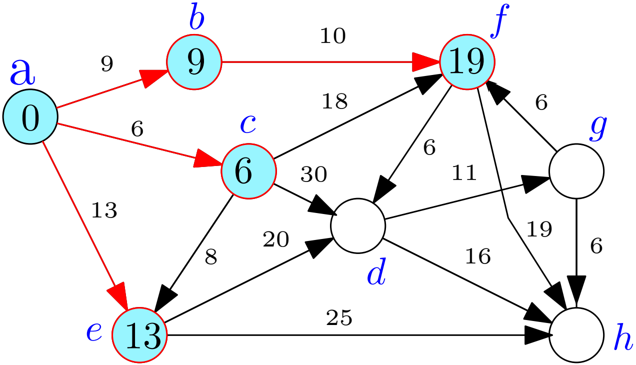

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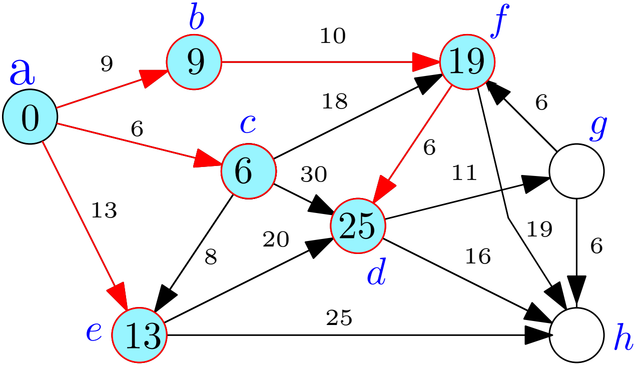

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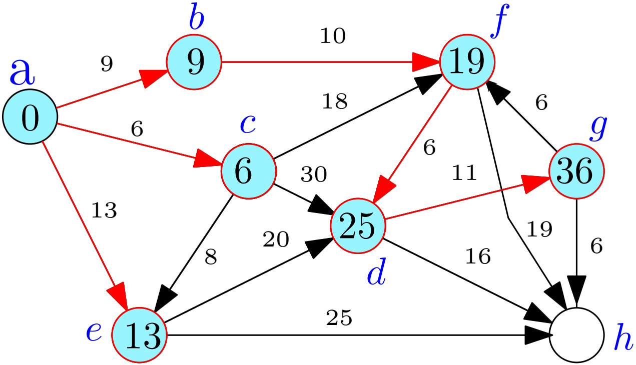

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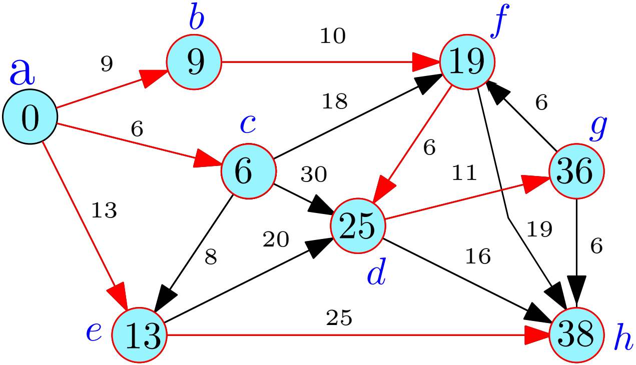

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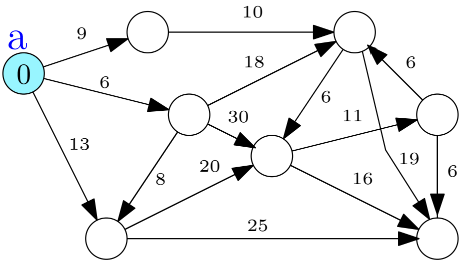

Corollary

The **i***th closest node is adjacent to* **S**.

Finding the ith closest node

1. $\mathbf{S}$ contains the $\mathbf{i-1}$ closest nodes to $\mathbf{s}$
2. Want to find the $\mathbf{i}$th closest node from $\mathbf{V-S}$.

1. For each $\mathbf{u \in V-S}$ let $\mathbf{P(s,u,S)}$ be a shortest path from $\mathbf{s}$ to $\mathbf{u}$ using only nodes in $\mathbf{S}$ as intermediate vertices.
2. Let $\mathbf{d'(s,u)}$ be the length of $\mathbf{P(s,u,S)}$

Observations: for each $\mathbf{u \in V-S}$,

1. $\mathrm{dist}(\mathbf{s,u}) \leq \mathbf{d'(s,u)}$ since we are constraining the paths
2. $\mathbf{d'(s,u)} = \mathbf{\min_{a \in S}}(\mathrm{dist}(\mathbf{s,a}) + \ell(\mathbf{a,u}))$ - Why?

Lemma

If $\mathbf{v}$ *is the* $\mathbf{i}$*th closest node to* $\mathbf{s}$*, then* $\mathbf{d'(s,v)} = \mathrm{dist}(\mathbf{s,v})$.

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Finding the $\mathbf{i}$th closest node

Lemma

Given:

1. S: *Set of* $\mathbf{i} - \mathbf{1}$ *closest nodes to* s.
2. $\mathbf{d}'(\mathbf{s}, \mathbf{u}) = \min_{\mathbf{x} \in \mathbf{S}}(\mathrm{dist}(\mathbf{s}, \mathbf{x}) + \ell(\mathbf{x}, \mathbf{u}))$

If $\mathbf{v}$ *is an* $\mathbf{i}$*th closest node to* $\mathbf{s}$*, then* $\mathbf{d}'(\mathbf{s}, \mathbf{v}) = \mathrm{dist}(\mathbf{s}, \mathbf{v})$.

Proof.

Let $\mathbf{v}$ be the $\mathbf{i}$th closest node to $\mathbf{s}$. Then there is a shortest path $\mathbf{P}$ from $\mathbf{s}$ to $\mathbf{v}$ that contains only nodes in $\mathbf{S}$ as intermediate nodes (see previous claim). Therefore $\mathbf{d}'(\mathbf{s}, \mathbf{v}) = \mathrm{dist}(\mathbf{s}, \mathbf{v})$. □

Finding the ith closest node

Lemma

If $\mathbf{v}$ *is an* $\mathbf{i}$*th closest node to* $\mathbf{s}$*, then* $\mathbf{d'(s,v)} = \mathrm{dist}(\mathbf{s},\mathbf{v})$.

Corollary

The $\mathbf{i}$*th closest node to* $\mathbf{s}$ *is the node* $\mathbf{v} \in \mathbf{V} - \mathbf{S}$ *such that* $\mathbf{d'(s,v)} = \mathbf{\min_{u \in V - S} d'(s,u)}$.

Proof.

For every node $\mathbf{u} \in \mathbf{V} - \mathbf{S}$, $\mathrm{dist}(\mathbf{s},\mathbf{u}) \leq \mathbf{d'(s,u)}$ and for the $\mathbf{i}$th closest node $\mathbf{v}$, $\mathrm{dist}(\mathbf{s},\mathbf{v}) = \mathbf{d'(s,v)}$. Moreover, $\mathrm{dist}(\mathbf{s},\mathbf{u}) \geq \mathrm{dist}(\mathbf{s},\mathbf{v})$ for each $\mathbf{u} \in \mathbf{V} - \mathbf{S}$. □

Algorithm

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Initialize for each node v:  dist(s, v) = ∞
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     using only S as intermediate nodes*)
    Let v be such that d'(s, v) = min_{u∈V−S} d'(s, u)
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    for each node u in V \ S do
        d'(s, u) ⇐ min_{a∈S} (dist(s, a) + ℓ(a, u))
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Correctness: By induction on i using previous lemmas.

Running time: $O(n \cdot (n + m))$ time.

1. n outer iterations. In each iteration, $d'(s, u)$ for each u by scanning all edges out of nodes in S; $O(m + n)$ time/iteration.

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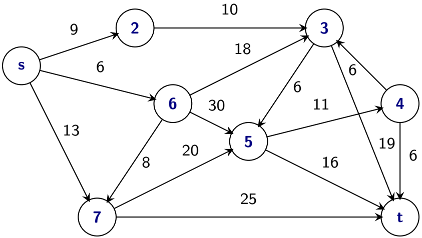
Example



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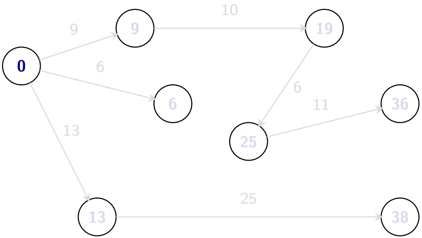

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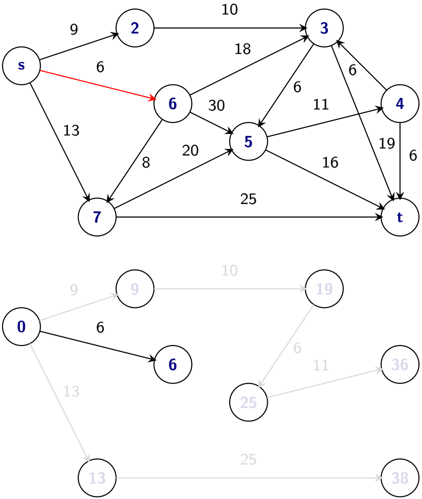

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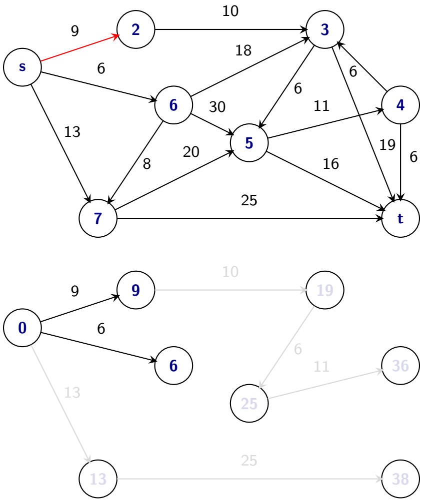

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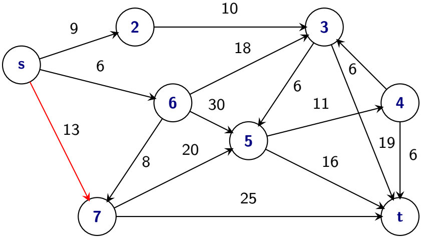
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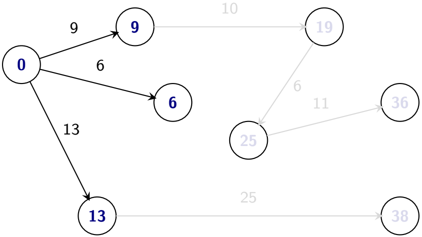
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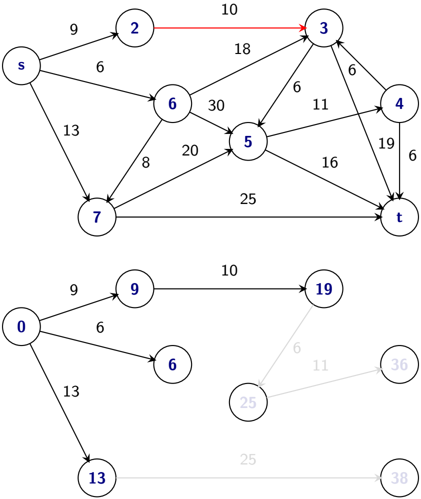
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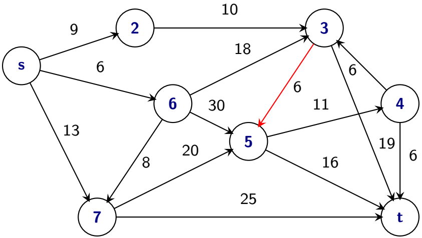

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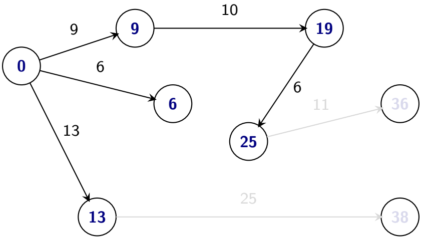

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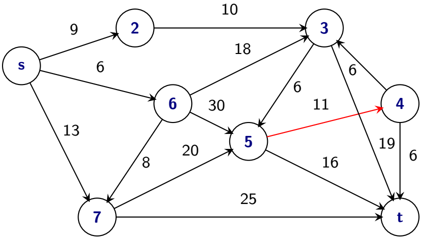

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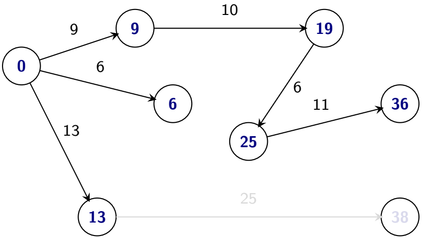

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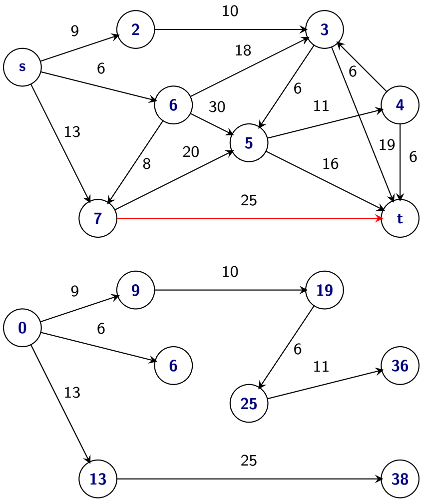

Improved Algorithm

1. Main work is to compute the $\mathbf{d'(s,u)}$ values in each iteration
2. $\mathbf{d'(s,u)}$ changes from iteration $\mathbf{i}$ to $\mathbf{i+1}$ only because of the node $\mathbf{v}$ that is added to $\mathbf{S}$ in iteration $\mathbf{i}$.

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Running time: $O(m + n^2)$ time.

1. $\mathbf{n}$ outer iterations and in each iteration following steps
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1. n outer iterations and in each iteration following steps
2. updating $d'(s, u)$ after v added takes $O(\deg(v))$ time so total work is $O(m)$ since a node enters S only once
3. Finding v from $d'(s, u)$ values is $O(n)$ time

Dijkstra's Algorithm

1. eliminate $d'(s,u)$ and let $\text{dist}(s,u)$ maintain it
2. update **dist** values after adding **v** by scanning edges out of **v**

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Priority Queues to maintain **dist** values for faster running time

1. Using heaps and standard priority queues: $O((m+n)\log n)$
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Priority Queues

Data structure to store a set $\mathbf{S}$ of $\mathbf{n}$ elements where each element $\mathbf{v} \in \mathbf{S}$ has an associated real/integer key $\mathbf{k(v)}$ such that the following operations:

1. **makePQ**: create an empty queue.
2. **findMin**: find the minimum key in $\mathbf{S}$.
3. **extractMin**: Remove $\mathbf{v} \in \mathbf{S}$ with smallest key and return it.
4. **insert**$(\mathbf{v}, \mathbf{k(v)})$: Add new element $\mathbf{v}$ with key $\mathbf{k(v)}$ to $\mathbf{S}$.
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All operations can be performed in $\mathbf{O(\log n)}$ time.
decreaseKey is implemented via **delete** and **insert**.

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Dijkstra's Algorithm using Priority Queues

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for each node u ≠ s do
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S ⇐ ∅
for i = 1 to |V| do
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```

Priority Queue operations:

1. $O(n)$ **insert** operations
2. $O(n)$ **extractMin** operations
3. $O(m)$ **decreaseKey** operations

Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

1. All operations can be done in $\mathbf{O(\log n)}$ time

Dijkstra's algorithm can be implemented in $\mathbf{O((n+m)\log n)}$ time.

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Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps

1. **extractMin**, **insert**, **delete**, **meld** in $\mathbf{O(\log n)}$ time
2. **decreaseKey** in $\mathbf{O(1)}$ *amortized* time: ℓ **decreaseKey** operations for $\ell \geq \mathbf{n}$ take *together* $\mathbf{O(\ell)}$ time
3. Relaxed Heaps: **decreaseKey** in $\mathbf{O(1)}$ worst case time but at the expense of **meld** (not necessary for Dijkstra's algorithm)

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2. Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)

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Shortest Path Tree

Dijkstra's algorithm finds the shortest path distances from s to **V**.
Question: How do we find the paths themselves?

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for each node u ≠ s do
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Shortest Path Tree

Lemma

The edge set $(\mathbf{u}, \mathrm{prev}(\mathbf{u}))$ *is the* reverse *of a shortest path tree rooted at* $\mathbf{s}$*. For each* $\mathbf{u}$*, the reverse of the path from* $\mathbf{u}$ *to* $\mathbf{s}$ *in the tree is a shortest path from* $\mathbf{s}$ *to* $\mathbf{u}$*.*

Proof Sketch.

1. The edge set $\{(\mathbf{u}, \mathrm{prev}(\mathbf{u})) \mid \mathbf{u} \in \mathbf{V}\}$ induces a directed in-tree rooted at $\mathbf{s}$ (Why?)
2. Use induction on $|\mathbf{S}|$ to argue that the tree is a shortest path tree for nodes in $\mathbf{V}$.

□

Shortest paths to s

Dijkstra's algorithm gives shortest paths from **s** to all nodes in **V**.
How do we find shortest paths from all of **V** to **s**?

1. In undirected graphs shortest path from **s** to **u** is a shortest path from **u** to **s** so there is no need to distinguish.
2. In directed graphs, use Dijkstra's algorithm in $\mathbf{G}^{\mathrm{rev}}$!

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