1. Let \((G, s, t)\) be a flow network with integer capacities. An edge \(e\) is upper-binding if increasing the capacity on \(e\) increases the value of the maximum flow in \(G\).

Describe and analyze an algorithm that finds all the upper-binding edges in the graph.

2. The NSA has established several monitoring stations around the country, each one conveniently hidden in the back of a Starbucks. Each station can monitor up to 42 cell-phone towers, but can only monitor cell-phone towers within a 20-mile radius. To ensure that every cell-phone call is recorded even if some stations malfunction, the NSA requires each cell-phone tower to be monitored by at least 3 different stations.

Suppose you know that there are \(n\) cell-phone towers and \(m\) monitoring stations, and you are given a function \(\text{Distance}(i, j)\) that returns the distance between the \(i\)-th tower and the \(j\)-th station in \(O(1)\) time. Describe and analyze an algorithm that either computes a valid assignment of cell-phone towers to monitoring stations, or reports correctly that there is no such assignment (in which case the NSA will build yet another Starbucks).

3. Let \(G = (V, E)\) be an arbitrary connected graph with weighted edges.

(a) Prove that for any partition of the vertices \(V\) into two disjoint subsets, the minimum spanning tree of \(G\) includes a minimum-weight edge among those with one endpoint in each subset.

(b) Prove that for any cycle in \(G\), the minimum spanning tree of \(G\) excludes the maximum-weight edge in that cycle.

4. A polygonal path is a sequence of line segments joined end-to-end; the endpoints of these line segments are called the vertices of the path. The length of a polygonal path is the sum of the lengths of its segments. A polygonal path with vertices \((x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\) is monotonically increasing if \(x_i < x_{i+1}\) and \(y_i < y_{i+1}\) for every index \(i\) – informally, each vertex of the path is above and to the right of its predecessor.

Suppose you are given a set \(S\) of \(n\) points in the plane, represented as two arrays \(X[1\ldots n]\) and \(Y[1\ldots n]\). Describe and analyze an algorithm to compute the length of the maximum-length monotonically increasing path with vertices in \(S\).

5. Suppose we are given an \(n\)-digit integer \(X\). Repeatedly remove one digit from either end of \(X\) (your choice) until no digits are left. The square-depth of \(X\) is the maximum number of perfect squares that you can see during this process.

Describe and analyze an algorithm to compute the square-depth of a given integer \(X\), represented as an array \(X[1\ldots n]\) of \(n\) decimal digits. Assume you have access to a subroutine \(\text{IsSquare}\) that determines whether a given \(k\)-digit number is a perfect square in \(O(k^2)\) time.

6. Consider a full binary tree of height \(h\). Starting from the root, flip a coin and go the left subtree with probability 1/2 (if you get a head), and to the right subtree with
probability 1/2 (if you get a tail). Continue to flip coins and take subtrees until you arrive at a leaf.

Define the random variable

$$X_h = \alpha^k,$$

where $k$ is the number of left subtrees taken on the walk down to the leaf. Prove that $E[X_h] = \left(\frac{1+\alpha}{2}\right)^h$. 