1. Consider the following statement.

   - Prove that every tree with n nodes has $n-1$ edges.

   A student in Cornfield University, learning induction from some badly-written template, sketch the following "proof":

   > When $n = 1$, the statement is clearly true. Assume that for $k \leq n$ the statement is true for any tree with k nodes. Let T be a tree with n nodes. By induction hypothesis T has $n-1$ edges. If we add one more node as a leaf to T and get a new tree T', the new tree T' has $n+1$ nodes and n edges. Therefore the statement is true for all n by induction.

   (a) Explain why the above proof is wrong.

   (b) Prove the statement correctly.

2. A *tournament* is a directed graph with exactly one directed edge between each pair of vertices. That is, for any vertices v and w, a tournament contains either an edge $v \to w$ or an edge $w \to v$, but not both. A *Hamiltonian path* in a directed graph G is a directed path that visits every vertex of G exactly once.

   Prove that every tournament contains a Hamiltonian path.

3. Consider the following sequence a_n: Set $a_1 = 2$, $a_2 = 3$, and define

   $$a_n = 1 + \min_{1<k<n} k \cdot a_{\lfloor \frac{n-1}{k} \rfloor} \qquad \text{for all } n \geq 3.$$

   Prove that $a_n \geq n$ for all positive integer n.

Here are some extra problems for you:

α. For any non-negative integer n, the $2^n \times 2^n$ checkerboard with an arbitrary block removed can be tiled using copies of the L-shaped triominos.

β. (a) Prove that every nonnegative integer n can be written as a sum of powers of 2. ("Write n in binary" is not a proof!)

(b) Prove that every nonnegative integer can be written as the sum of distinct, non-consecutive Fibonacci numbers. That is, if the Fibonacci number F_i appears in the sum, it appears exactly once, and its neighbors F_{i-1} and F_{i+1} do not appear at all. For example, $54 = F_9 + F_7 + F_5 + F_3$.

γ. Prove the *AM-GM inequality* using induction:

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}.$$

[Hint: Handle the case for $n = 2$ separately. Try to prove the case when n is a power of 2 first.]