Polynomial Time Reductions

Lecture 20

April 9, 2013

Part I

Introduction to Reductions

Reductions

A reduction from Problem **X** to Problem **Y** means (informally) that if we have an algorithm for Problem **Y**, we can use it to find an algorithm for Problem **X**.

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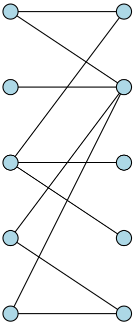
Also, the right reductions might win you a million dollars!

Example 1: Bipartite Matching and Flows

How do we solve the Bipartite Matching Problem?

Given a bipartite graph $G = (U \cup V, E)$ and number k, does G have a matching of size $\geq k$?



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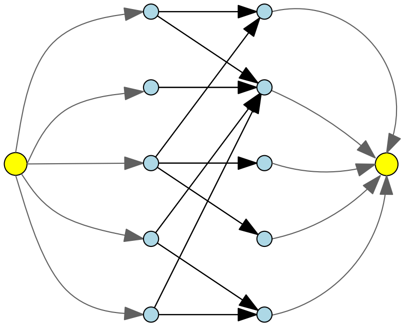

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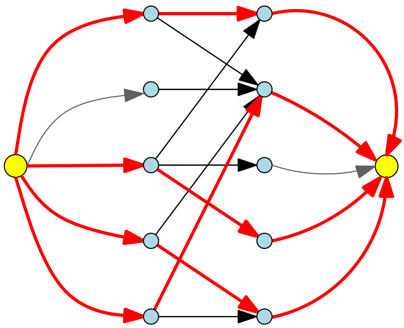

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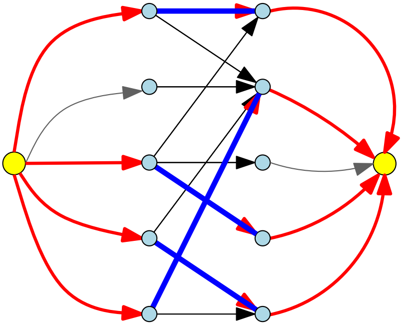

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Types of Problems

Decision, Search, and Optimization

1. **Decision problem**. Example: given $\mathbf{n}$, is $\mathbf{n}$ prime?.
2. **Search problem**. Example: given $\mathbf{n}$, find a factor of $\mathbf{n}$ if it exists.
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For max flow...

Problem (Max-Flow optimization version)

Given an instance G of network flow, find the maximum flow between **s** *and* **t**.

Problem (Max-Flow decision version)

Given an instance G of network flow and a parameter **K**, *is there a flow in G, from* **s** *to* **t**, *of value at least* **K**?

While using reductions and comparing problems, we typically work with the decision versions. Decision problems have Yes/No answers. This makes them easy to work with.

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Problems vs Instances

1. A problem Π consists of an **infinite** collection of inputs $\{I_1, I_2, \ldots, \}$. Each input is referred to as an instance.
2. The size of an instance I is the number of bits in its representation.
3. For an instance I, $sol(I)$ is a set of feasible solutions to I.
4. For optimization problems each solution $s \in sol(I)$ has an associated value.

Examples

Example

An instance of **Bipartite Matching** is a bipartite graph, and an integer $\mathbf{k}$. The solution to this instance is "YES" if the graph has a matching of size $\geq \mathbf{k}$, and "NO" otherwise.

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An instance of **Max-Flow** is a graph $\mathbf{G}$ with edge-capacities, two vertices $\mathbf{s}, \mathbf{t}$, and an integer $\mathbf{k}$. The solution to this instance is "YES" if there is a flow from $\mathbf{s}$ to $\mathbf{t}$ of value $\geq \mathbf{k}$, else 'NO".

What is an algorithm for a decision Problem $\mathbf{X}$?

It takes as input an instance of $\mathbf{X}$, and outputs either "YES" or "NO".

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Encoding an instance into a string

1. **I**; Instance of some problem.
2. **I** can be fully and precisely described (say in a text file).
3. Resulting text file is a binary string.
4. $\implies$ Any input can be interpreted as a binary string **S**.
5. ... Running time of algorithm: Function of length of **S** (i.e., **n**).

Decision Problems and Languages

1. A finite alphabet Σ. Σ^* is set of all finite strings on Σ.
2. A language L is simply a subset of Σ^*; a set of strings.

For every language L there is an associated decision problem Π_L and conversely, for every decision problem Π there is an associated language L_Π.

1. Given L, Π_L is the following decision problem: Given $x \in \Sigma^*$, is $x \in L$? Each string in Σ^* is an instance of Π_L and L is the set of instances for which the answer is YES.
2. Given Π the associated language is

$$L_\Pi = \left\{ I \;\middle|\; I \text{ is an instance of } \Pi \text{ for which answer is YES} \right\}.$$

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Example

1. The decision problem **Primality**, and the language

$$\mathbf{L} = \left\{ \#\mathbf{p} \;\middle|\; \mathbf{p} \text{ is a prime number} \right\}.$$

   Here $\#\mathbf{p}$ is the string in base $\mathbf{10}$ representing $\mathbf{p}$.

2. **Bipartite** (is given graph is bipartite. The language is

$$\mathbf{L} = \left\{ \boldsymbol{\mathcal{S}}(\mathsf{G}) \;\middle|\; \mathsf{G} \text{ is a bipartite graph} \right\}.$$

   Here $\boldsymbol{\mathcal{S}}(\mathsf{G})$ is the string encoding the graph G.

Reductions, revised.

For decision problems **X, Y**, a **reduction from X to Y** is:

1. An algorithm ...
2. Input: $\mathbf{I_X}$, an instance of **X**.
3. Output: $\mathbf{I_Y}$ an instance of **Y**.
4. Such that:

$$\boxed{\mathbf{I_Y} \text{ is YES instance of } \mathbf{Y}} \iff \boxed{\mathbf{I_X} \text{ is YES instance of } \mathbf{X}}$$

There are other kinds of reductions.

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Using reductions to solve problems

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2. $\mathcal{A}_{\mathbf{Y}}$: algorithm for $\mathbf{Y}$:
3. $\implies$ New algorithm for $\mathbf{X}$:

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A_X(I_X):
        // I_X: instance of X.
        I_Y <= R(I_X)
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If $\mathcal{R}$ and $\mathcal{A}_{\mathbf{Y}}$ polynomial-time $\implies$ $\mathcal{A}_{\mathbf{X}}$ polynomial-time.

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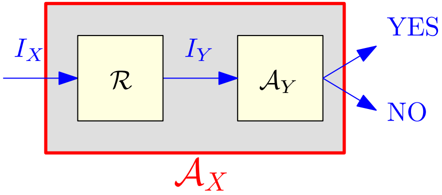

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Comparing Problems

1. "Problem **X** is no harder to solve than Problem **Y**".
2. If Problem **X** reduces to Problem **Y** (we write **X** $\leq$ **Y**), then **X** cannot be harder to solve than **Y**.
3. **Bipartite Matching** $\leq$ **Max-Flow**.
   Bipartite Matching cannot be harder than **Max-Flow**.
4. Equivalently,
   Max-Flow is at least as hard as **Bipartite Matching**.
5. **X** $\leq$ **Y**:
   1. **X** is no harder than **Y**, or
   2. **Y** is at least as hard as **X**.

Part II

Examples of Reductions

Independent Sets and Cliques

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1. **independent set**: no two vertices of $\mathbf{V}'$ connected by an edge.

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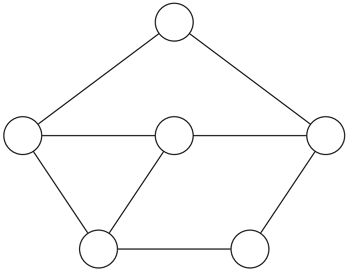

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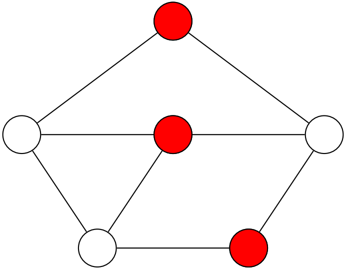

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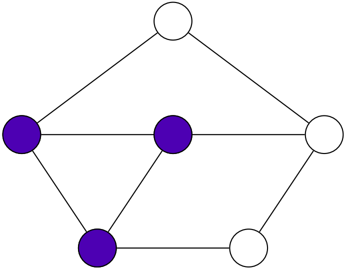

The Independent Set and Clique Problems

Problem: Independent Set

Instance: A graph G and an integer $\mathbf{k}$.
Question: Does G has an independent set of size $\geq \mathbf{k}$?

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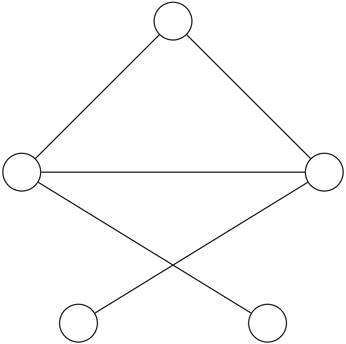

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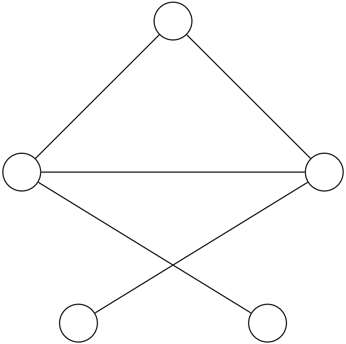
We use $\overline{\mathbf{G}}$ and **k** as the instance of **Clique**.



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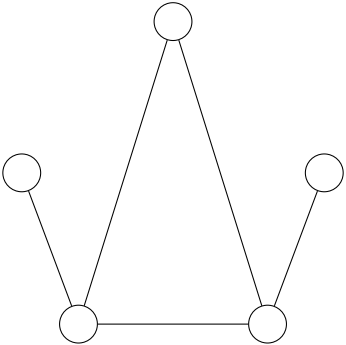

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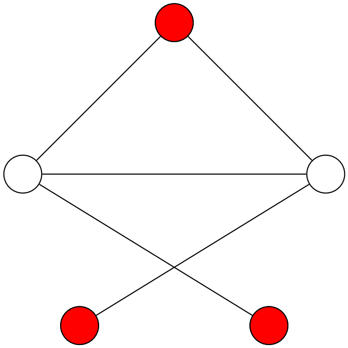

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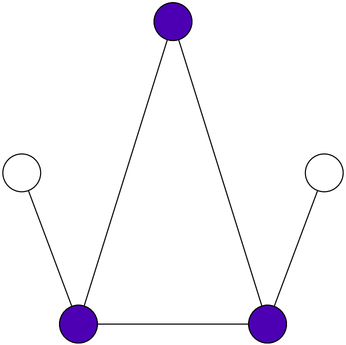

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   What does this mean?
2. If have an algorithm for **Clique**, then we have an algorithm for **Independent Set**.
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DFAs (Remember 373?) are automata that accept regular languages. NFAs are the same, except that they are non-deterministic, while DFAs are deterministic.

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(How long does this take?)
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That is, $\mathbf{L(M)} = \mathbf{\Sigma}^*$, the set of all strings.

Problem (DFA universality)

Input: *A DFA* $\mathbf{M}$*.*
Goal: *Is* $\mathbf{M}$ *universal?*

How do we solve **DFA Universality**?
We check if $\mathbf{M}$ has *any* reachable non-final state.
Alternatively, minimize $\mathbf{M}$ to obtain $\mathbf{M'}$ and see if $\mathbf{M'}$ has a single state which is an accepting state.

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Alternatively, minimize **M** to obtain **M′** and see if **M′** has a single state which is an accepting state.

NFA Universality

An NFA **N** is said to be universal if it accepts every string. That is, $\mathbf{L(N) = \Sigma^*}$, the set of all strings.

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We say that an algorithm is **efficient** if it runs in polynomial-time.

To find efficient algorithms for problems, we are only interested in polynomial-time reductions. Reductions that take longer are not useful.

If we have a polynomial-time reduction from problem $\mathbf{X}$ to problem $\mathbf{Y}$ (we write $\mathbf{X} \leq_P \mathbf{Y}$), and a poly-time algorithm $\mathcal{A}_\mathbf{Y}$ for $\mathbf{Y}$, we have a polynomial-time/efficient algorithm for $\mathbf{X}$.

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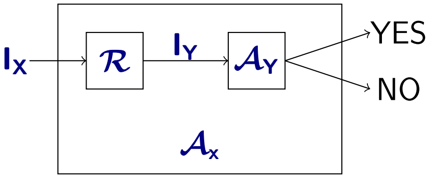

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Proposition

If $\mathbf{X} \leq_{\mathbf{P}} \mathbf{Y}$ *then a polynomial time algorithm for* $\mathbf{Y}$ *implies a polynomial time algorithm for* $\mathbf{X}$.

Such a reduction is called a **Karp reduction**. Most reductions we will need are Karp reductions.

Polynomial-time reductions and hardness

For decision problems **X** and **Y**, if **X** $\leq_P$ **Y**, and **Y** has an efficient algorithm, **X** has an efficient algorithm.

If you believe that **Independent Set** does not have an efficient algorithm, why should you believe the same of **Clique**?

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Polynomial-time reductions and instance sizes

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Let $\mathcal{R}$ be a polynomial-time reduction from $\mathbf{X}$ to $\mathbf{Y}$. Then for any instance $\mathbf{I_X}$ of $\mathbf{X}$, the size of the instance $\mathbf{I_Y}$ of $\mathbf{Y}$ produced from $\mathbf{I_X}$ by $\mathcal{R}$ is polynomial in the size of $\mathbf{I_X}$.

Proof.

$\mathcal{R}$ is a polynomial-time algorithm and hence on input $\mathbf{I_X}$ of size $|\mathbf{I_X}|$ it runs in time $\mathbf{p}(|\mathbf{I_X}|)$ for some polynomial $\mathbf{p}()$.

$\mathbf{I_Y}$ is the output of $\mathcal{R}$ on input $\mathbf{I_X}$.

$\mathcal{R}$ can write at most $\mathbf{p}(|\mathbf{I_X}|)$ bits and hence $|\mathbf{I_Y}| \leq \mathbf{p}(|\mathbf{I_X}|)$. □

Note: Converse is not true. A reduction need not be polynomial-time even if output of reduction is of size polynomial in its input.

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Transitivity of Reductions

Proposition

$\mathbf{X} \leq_{\mathbf{P}} \mathbf{Y}$ *and* $\mathbf{Y} \leq_{\mathbf{P}} \mathbf{Z}$ *implies that* $\mathbf{X} \leq_{\mathbf{P}} \mathbf{Z}$.

Note: $\mathbf{X} \leq_{\mathbf{P}} \mathbf{Y}$ does not imply that $\mathbf{Y} \leq_{\mathbf{P}} \mathbf{X}$ and hence it is very important to know the FROM and TO in a reduction.

To prove $\mathbf{X} \leq_{\mathbf{P}} \mathbf{Y}$ you need to show a reduction FROM $\mathbf{X}$ TO $\mathbf{Y}$
In other words show that an algorithm for $\mathbf{Y}$ implies an algorithm for $\mathbf{X}$.

Vertex Cover

Given a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, a set of vertices $\mathbf{S}$ is:

1. A **vertex cover** if every $\mathbf{e} \in \mathbf{E}$ has at least one endpoint in $\mathbf{S}$.

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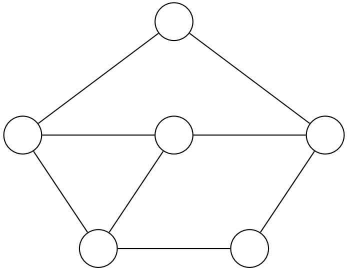

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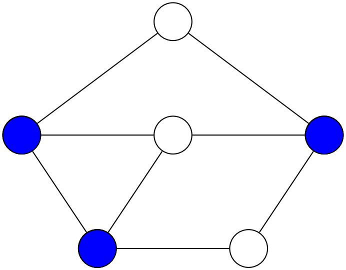

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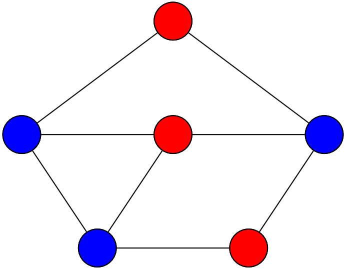

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Problem (**Vertex Cover**)

Input: *A graph G and integer $\mathbf{k}$.*
Goal: *Is there a vertex cover of size $\leq \mathbf{k}$ in G?*

Can we relate **Independent Set** and **Vertex Cover**?

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Relationship between...

Vertex Cover and Independent Set

Proposition

Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ *be a graph.* $\mathbf{S}$ *is an independent set if and only if* $\mathbf{V} \setminus \mathbf{S}$ *is a vertex cover.*

Proof.

($\Rightarrow$) Let $\mathbf{S}$ be an independent set

1. Consider any edge $\mathbf{uv} \in \mathbf{E}$.
2. Since $\mathbf{S}$ is an independent set, either $\mathbf{u} \not\in \mathbf{S}$ or $\mathbf{v} \not\in \mathbf{S}$.
3. Thus, either $\mathbf{u} \in \mathbf{V} \setminus \mathbf{S}$ or $\mathbf{v} \in \mathbf{V} \setminus \mathbf{S}$.
4. $\mathbf{V} \setminus \mathbf{S}$ is a vertex cover.

($\Leftarrow$) Let $\mathbf{V} \setminus \mathbf{S}$ be some vertex cover:

1. Consider $\mathbf{u}, \mathbf{v} \in \mathbf{S}$
2. $\mathbf{uv}$ is not an edge of G, as otherwise $\mathbf{V} \setminus \mathbf{S}$ does not cover $\mathbf{uv}$.
3. $\implies$ $\mathbf{S}$ is thus an independent set. □

Independent Set $\leq_P$ Vertex Cover

1. G: graph with n vertices, and an integer k be an instance of the **Independent Set** problem.
2. G has an independent set of size $\geq k$ iff G has a vertex cover of size $\leq n - k$
3. (G, k) is an instance of **Independent Set** , and $(G, n - k)$ is an instance of **Vertex Cover** with the same answer.
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A problem of Languages

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Due to budget cuts, you can only afford to keep **k** translators on your payroll. Can you do this, while still ensuring that there is someone who speaks every language in **U**?

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Problem (Set Cover)

Input: *Given a set* $\mathbf{U}$ *of* $\mathbf{n}$ *elements, a collection* $\mathbf{S_1, S_2, \ldots S_m}$ *of subsets of* $\mathbf{U}$, *and an integer* $\mathbf{k}$.

Goal: *Is there a collection of at most* $\mathbf{k}$ *of these sets* $\mathbf{S_i}$ *whose union is equal to* $\mathbf{U}$?

Example

Let $\mathbf{U} = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathbf{k} = 2$ with

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Observe that G has vertex cover of size k if and only if $U, \{S_v\}_{v \in V}$ has a set cover of size k. (Exercise: Prove this.)

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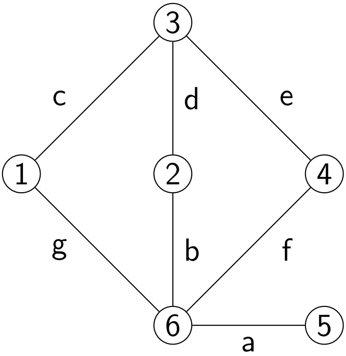

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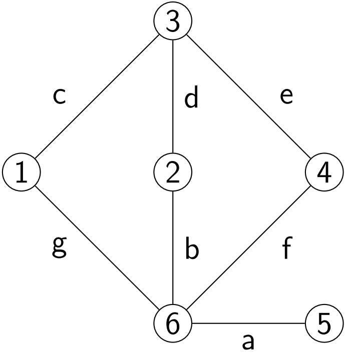

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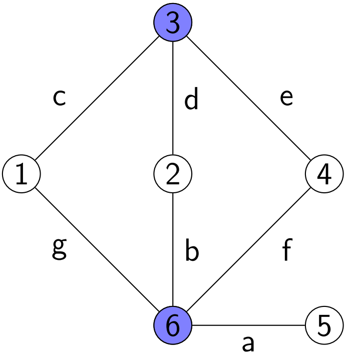

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Proving Reductions

To prove that $\mathbf{X} \leq_P \mathbf{Y}$ you need to give an algorithm $\mathcal{A}$ that:

1. Transforms an instance $\mathbf{I_X}$ of $\mathbf{X}$ into an instance $\mathbf{I_Y}$ of $\mathbf{Y}$.
2. Satisfies the property that answer to $\mathbf{I_X}$ is YES iff $\mathbf{I_Y}$ is YES.
    1. typical easy direction to prove: answer to $\mathbf{I_Y}$ is YES if answer to $\mathbf{I_X}$ is YES
    2. typical difficult direction to prove: answer to $\mathbf{I_X}$ is YES if answer to $\mathbf{I_Y}$ is YES (equivalently answer to $\mathbf{I_X}$ is NO if answer to $\mathbf{I_Y}$ is NO).
3. Runs in **polynomial** time.

Example of incorrect reduction proof

Try proving **Matching** $\leq_P$ **Bipartite Matching** via following reduction:

1. Given graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ obtain a bipartite graph $\mathbf{G}' = (\mathbf{V}', \mathbf{E}')$ as follows.
    1. Let $\mathbf{V_1} = \{\mathbf{u_1} \mid \mathbf{u} \in \mathbf{V}\}$ and $\mathbf{V_2} = \{\mathbf{u_2} \mid \mathbf{u} \in \mathbf{V}\}$. We set $\mathbf{V}' = \mathbf{V_1} \cup \mathbf{V_2}$ (that is, we make two copies of $\mathbf{V}$)
    2. $\mathbf{E}' = \left\{ \mathbf{u_1 v_2} \;\middle|\; \mathbf{u} \neq \mathbf{v} \text{ and } \mathbf{uv} \in \mathbf{E} \right\}$
2. Given $\mathbf{G}$ and integer $\mathbf{k}$ the reduction outputs $\mathbf{G}'$ and $\mathbf{k}$.

Example

"Proof"

Claim

Reduction is a poly-time algorithm. If $\mathbf{G}$ *has a matching of size* $\mathbf{k}$ *then* $\mathbf{G'}$ *has a matching of size* $\mathbf{k}$.

Proof.

Exercise. □

Claim

If $\mathbf{G'}$ *has a matching of size* $\mathbf{k}$ *then* $\mathbf{G}$ *has a matching of size* $\mathbf{k}$.

Incorrect! Why? Vertex $\mathbf{u} \in \mathbf{V}$ has two copies $\mathbf{u_1}$ and $\mathbf{u_2}$ in $\mathbf{G'}$. A matching in $\mathbf{G'}$ may use both copies!

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Reduction is a poly-time algorithm. If $\mathbf{G}$ has a matching of size $\mathbf{k}$ then $\mathbf{G'}$ has a matching of size $\mathbf{k}$.

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Summary

We looked at polynomial-time reductions.

Using polynomial-time reductions

1. If $\mathbf{X} \leq_{\mathbf{P}} \mathbf{Y}$, and there is no efficient algorithm for $\mathbf{X}$, there is no efficient algorithm for $\mathbf{Y}$.

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We looked at some examples of reductions between **Independent Set**, **Clique**, **Vertex Cover**, and **Set Cover**.

Notes

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