

# Chapter 18

## Applications of Network Flows

CS 473: Fundamental Algorithms, Spring 2013

March 28, 2013

### 18.1 Important Properties of Flows

#### 18.1.0.1 Network Flow: Facts to Remember

Flow network: directed graph  $G$ , capacities  $c$ , source  $s$ , sink  $t$ .

(A) Maximum  $s$ - $t$  flow can be computed:

- (A) Using Ford-Fulkerson algorithm in  $O(mC)$  time when capacities are integral and  $C$  is an upper bound on the flow.
- (B) Using variant of algorithm, in  $O(m^2 \log C)$  time, when capacities are integral. (Polynomial time.)
- (C) Using Edmonds-Karp algorithm, in  $O(m^2n)$  time, when capacities are rational (strongly polynomial time algorithm).

#### 18.1.1 Network Flow

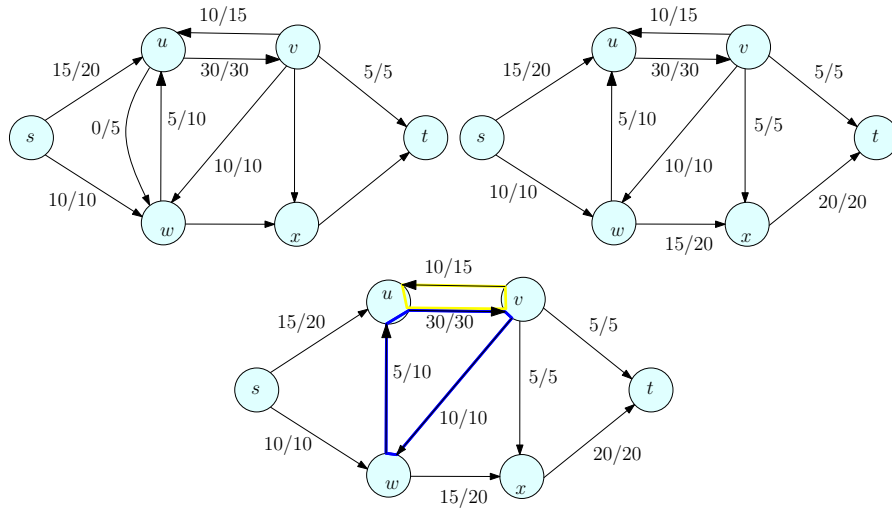
##### 18.1.1.1 Even more facts to remember

- (A) If capacities are integral then there is a maximum flow that is integral and above algorithms give an integral max flow. This is known as **integrality of flow**.
- (B) Given a flow of value  $v$ , can decompose into  $O(m+n)$  flow paths of same total value  $v$ . Integral flow implies integral flow on paths.
- (C) Maximum flow is equal to the minimum cut and minimum cut can be found in  $O(m+n)$  time given any maximum flow.

##### 18.1.1.2 Paths, Cycles and Acyclicity of Flows

**Definition 18.1.1.** Given a flow network  $G = (V, E)$  and a flow  $f : E \rightarrow \mathbb{R}^{\geq 0}$  on the edges, the **support** of  $f$  is the set of edges  $E' \subseteq E$  with non-zero flow on them. That is,  $E' = \{e \in E \mid f(e) > 0\}$ .

**Question:** Given a flow  $f$ , can there be cycles in its support?



### 18.1.1.3 Acyclicity of Flows

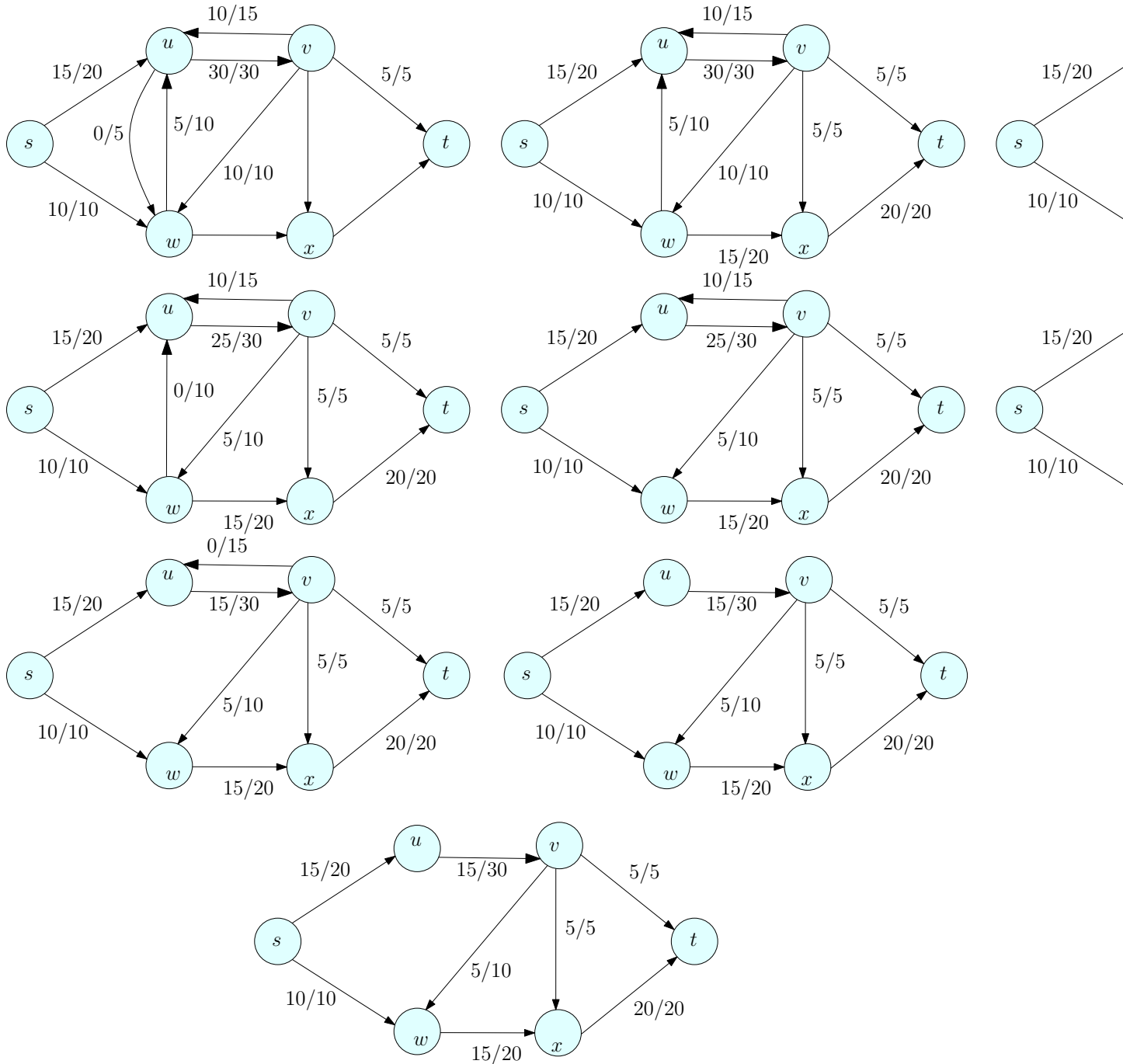
**Proposition 18.1.2.** *In any flow network, if  $f$  is a flow then there is another flow  $f'$  such that the support of  $f'$  is an acyclic graph and  $v(f') = v(f)$ . Further if  $f$  is an integral flow then so is  $f'$ .*

*Proof:*

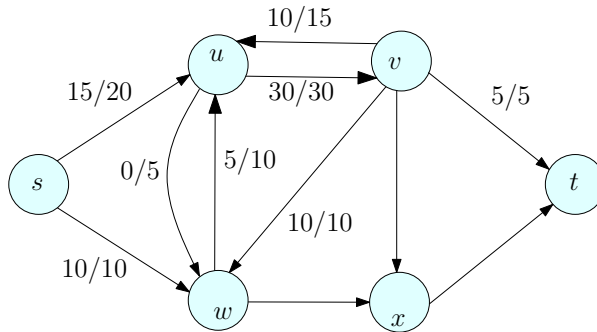
- (A)  $E' = \{e \in E \mid f(e) > 0\}$ , support of  $f$ .
- (B) Suppose there is a directed cycle  $C$  in  $E'$
- (C) Let  $e'$  be the edge in  $C$  with least amount of flow
- (D) For each  $e \in C$ , reduce flow by  $f(e')$ . Remains a flow. Why?
- (E) Flow on  $e'$  is reduced to 0.
- (F) Claim: Flow value from  $s$  to  $t$  does not change. Why?
- (G) Iterate until no cycles

■

### 18.1.1.4 Example



Throw away edge with no flow on it  
 Find a cycle in the support/flow  
 Reduce flow on cycle as much as possible  
 Throw away edge with no flow on it  
 Find a cycle in the support/flow  
 Reduce flow on cycle as much as possible  
 Throw away edge with no flow on it  
 Viola!!!  
 An equivalent flow with no cycles in it. Original flow:



### 18.1.1.5 Flow Decomposition

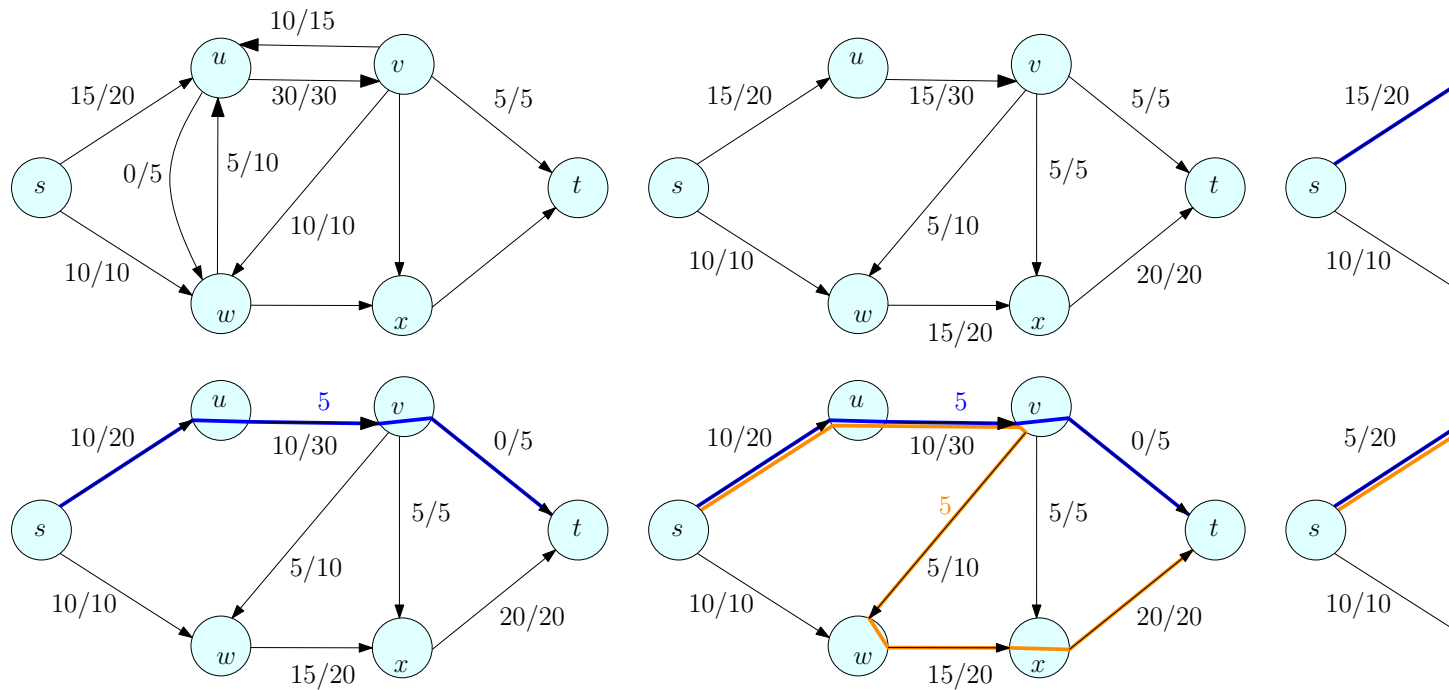
**Lemma 18.1.3.** Given an edge based flow  $f : E \rightarrow \mathbb{R}^{\geq 0}$ , there exists a collection of paths  $\mathcal{P}$  and cycles  $\mathcal{C}$  and an assignment of flow to them  $f' : \mathcal{P} \cup \mathcal{C} \rightarrow \mathbb{R}^{\geq 0}$  such that:

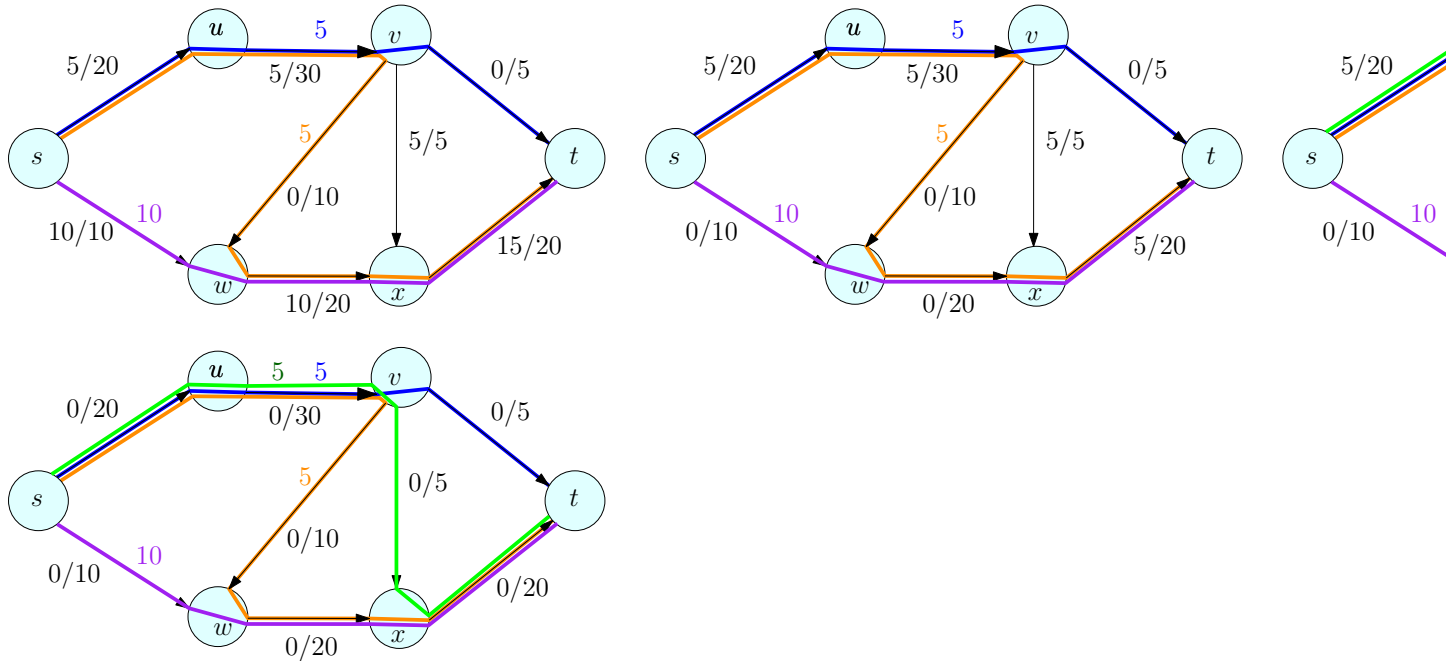
- (A)  $|\mathcal{P} \cup \mathcal{C}| \leq m$
- (B) for each  $e \in E$ ,  $\sum_{P \in \mathcal{P}: e \in P} f'(P) + \sum_{C \in \mathcal{C}: e \in C} f'(C) = f(e)$
- (C)  $v(f) = \sum_{P \in \mathcal{P}} f'(P)$ .
- (D) if  $f$  is integral then so are  $f'(P)$  and  $f'(C)$  for all  $P$  and  $C$

*Proof:*[Proof Idea]

- (A) Remove all cycles as in previous proposition.
- (B) Next, decompose into paths as in previous lecture.
- (C) Exercise: verify claims.

### 18.1.1.6 Example





Find cycles as shown before  
 Find a source to sink path, and push max flow along it (5 unites)  
 Compute remaining flow  
 Find a source to sink path, and push max flow along it (5 unites). Edges with 0 flow on them can not be used as they are no longer in the support of the flow.  
 Compute remaining flow  
 Find a source to sink path, and push max flow along it (10 unites). Compute remaining flow  
 Find a source to sink path, and push max flow along it (5 unites). Compute remaining flow  
 No flow remains in the graph. We fully decomposed the flow into flow on paths. Together with the cycles, we get a decomposition of the original flow into  $m$  flows on paths and cycles.

### 18.1.1.7 Flow Decomposition

**Lemma 18.1.4.** Given an edge based flow  $f : E \rightarrow \mathbb{R}^{\geq 0}$ , there exists a collection of paths  $\mathcal{P}$  and cycles  $\mathcal{C}$  and an assignment of flow to them  $f' : \mathcal{P} \cup \mathcal{C} \rightarrow \mathbb{R}^{\geq 0}$  such that:

- (A)  $|\mathcal{P} \cup \mathcal{C}| \leq m$
- (B) for each  $e \in E$ ,  $\sum_{P \in \mathcal{P}: e \in P} f'(P) + \sum_{C \in \mathcal{C}: e \in C} f'(C) = f(e)$
- (C)  $v(f) = \sum_{P \in \mathcal{P}} f'(P)$ .
- (D) if  $f$  is integral then so are  $f'(P)$  and  $f'(C)$  for all  $P$  and  $C$ .

Above flow decomposition can be computed in  $O(m^2)$  time.

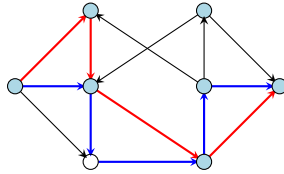
## 18.2 Network Flow Applications I

### 18.3 Edge Disjoint Paths

#### 18.3.1 Directed Graphs

##### 18.3.1.1 Edge-Disjoint Paths in Directed Graphs

Definition 18.3.1.



A set of paths is **edge disjoint** if no two paths share an edge.

**Problem** Given a directed graph with two special vertices  $s$  and  $t$ , find the *maximum* number of edge disjoint paths from  $s$  to  $t$ . **Applications:** Fault tolerance in routing — edges/nodes in networks can fail. Disjoint paths allow for planning backup routes in case of failures.

#### 18.3.2 Reduction to Max-Flow

##### 18.3.2.1 Reduction to Max-Flow

**Problem** Given a directed graph  $G$  with two special vertices  $s$  and  $t$ , find the maximum number of edge disjoint paths from  $s$  to  $t$ . **Reduction** Consider  $G$  as a flow network with edge capacities 1, and compute max-flow.

##### 18.3.2.2 Correctness of Reduction

**Lemma 18.3.2.** *If  $G$  has  $k$  edge disjoint paths  $P_1, P_2, \dots, P_k$  then there is an  $s$ - $t$  flow of value  $k$  in  $G$ .*

*Proof:* Set  $f(e) = 1$  if  $e$  belongs to one of the paths  $P_1, P_2, \dots, P_k$ ; other-wise set  $f(e) = 0$ . This defines a flow of value  $k$ . ■

##### 18.3.2.3 Correctness of Reduction

**Lemma 18.3.3.** *If  $G$  has a flow of value  $k$  then there are  $k$  edge disjoint paths between  $s$  and  $t$ .*

*Proof:*

- (A) Capacities are all 1 and hence there is integer flow of value  $k$ , that is  $f(e) = 0$  or  $f(e) = 1$  for each  $e$ .
- (B) Decompose flow into paths.
- (C) Flow on each path is either 1 or 0.
- (D) Hence there are  $k$  paths  $P_1, P_2, \dots, P_k$  with flow of 1 each.
- (E) Paths are edge-disjoint since capacities are 1. ■

### 18.3.2.4 Running Time

**Theorem 18.3.4.** *The number of edge disjoint paths in  $G$  can be found in  $O(mn)$  time.*

*Proof:*

- (A) Set capacities of edges in  $G$  to 1.
- (B) Run Ford-Fulkerson algorithm.
- (C) Maximum value of flow is  $n$  and hence run-time is  $O(nm)$ .
- (D) Decompose flow into  $k$  paths ( $k \leq n$ ).  
Takes  $O(k \times m) = O(km) = O(mn)$  time.

■

Remark The algorithm also computes a set of edge-disjoint paths realizing this optimal solution.

## 18.3.3 Menger's Theorem

### 18.3.3.1 Menger's Theorem

**Theorem 18.3.5 (Menger [1927]).** *Let  $G$  be a directed graph. The minimum number of edges whose removal disconnects  $s$  from  $t$  (the minimum-cut between  $s$  and  $t$ ) is equal to the maximum number of edge-disjoint paths in  $G$  between  $s$  and  $t$ .*

*Proof:* Maxflow-mincut theorem and integrality of flow.

■

Menger proved his theorem before Maxflow-Mincut theorem! Maxflow-Mincut theorem is a generalization of Menger's theorem to capacitated graphs.

## 18.3.4 Undirected Graphs

### 18.3.4.1 Edge Disjoint Paths in Undirected Graphs

Problem Given an **undirected** graph  $G$ , find the maximum number of edge disjoint paths in  $G$

Reduction:

- (A) create **directed** graph  $H$  by adding directed edges  $(u, v)$  and  $(v, u)$  for each edge  $uv$  in  $G$ .
- (B) compute maximum  $s$ - $t$  flow in  $H$ .

**Problem:** Both edges  $(u, v)$  and  $(v, u)$  may have non-zero flow!

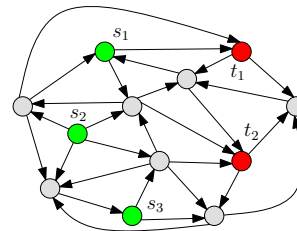
**Not a Problem!** Can assume maximum flow in  $H$  is acyclic and hence cannot have non-zero flow on both  $(u, v)$  and  $(v, u)$ . Reduction works. See book for more details.

## 18.4 Multiple Sources and Sinks

### 18.4.0.2 Multiple Sources and Sinks

Input:

- (A) A directed graph  $G$  with edge capacities  $c(e)$ .
- (B) Source nodes  $s_1, s_2, \dots, s_k$ .
- (C) Sink nodes  $t_1, t_2, \dots, t_\ell$ .
- (D) Sources and sinks are *disjoint*.



**Maximum Flow:** Send as much flow as possible from the sources to the sinks. *Sinks don't care which source they get flow from.*

**Minimum Cut:** Find a minimum capacity set of edge  $E'$  such that removing  $E'$  disconnects every source from every sink.

### 18.4.0.3 Multiple Sources and Sinks: Formal Definition

Input:

- (A) A directed graph  $G$  with edge capacities  $c(e)$ .
- (B) Source nodes  $s_1, s_2, \dots, s_k$ .
- (C) Sink nodes  $t_1, t_2, \dots, t_\ell$ .
- (D) Sources and sinks are *disjoint*.

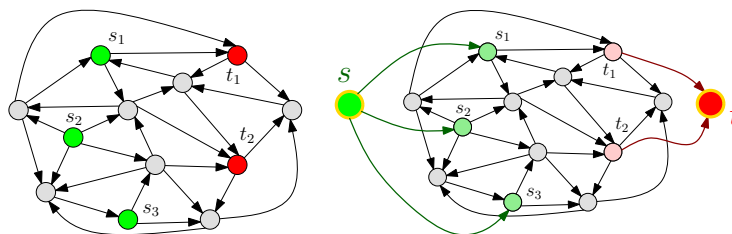
A function  $f : E \rightarrow \mathbb{R}^{\geq 0}$  is a **flow** if:

- (A) For each  $e \in E$ ,  $f(e) \leq c(e)$ , and
- (B) for each  $v$  which is not a source or a sink  $f^{\text{in}}(v) = f^{\text{out}}(v)$ .

**Goal:**  $\max \sum_{i=1}^k (f^{\text{out}}(s_i) - f^{\text{in}}(s_i))$ , that is, flow out of sources.

### 18.4.0.4 Reduction to Single-Source Single-Sink

- (A) Add a *source* node  $s$  and a sink node  $t$ .
- (B) Add edges  $(s, s_1), (s, s_2), \dots, (s, s_k)$ .
- (C) Add edges  $(t_1, t), (t_2, t), \dots, (t_\ell, t)$ .
- (D) Set the capacity of the new edges to be  $\infty$ .

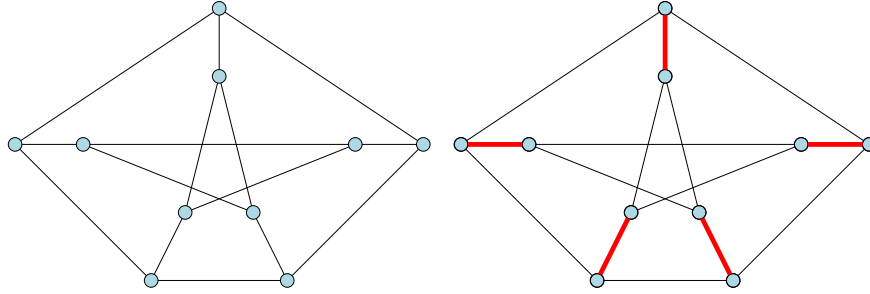


### 18.4.0.5 Supplies and Demands

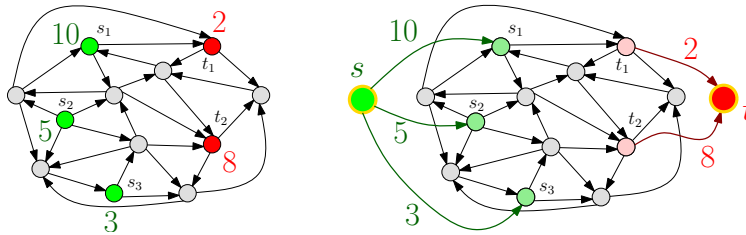
A further generalization:

- (A) source  $s_i$  has a supply of  $S_i \geq 0$
- (B) since  $t_j$  has a demand of  $D_j \geq 0$  units





**Question:** is there a flow from source to sinks such that supplies are not exceeded and demands are met? Formally we have the additional constraints that  $f^{\text{out}}(s_i) - f^{\text{in}}(s_i) \leq S_i$  for each source  $s_i$  and  $f^{\text{in}}(t_j) - f^{\text{out}}(t_j) \geq D_j$  for each sink  $t_j$ .



## 18.5 Bipartite Matching

### 18.5.1 Definitions

#### 18.5.1.1 Matching

**Problem 18.5.1 (Matching).**

**Input:** Given a (undirected) graph  $G = (V, E)$ .

**Goal:** Find a matching of maximum cardinality.

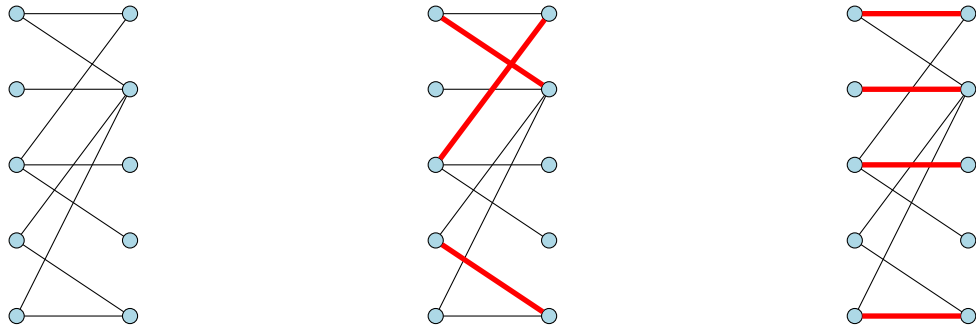
(A) A matching is  $M \subseteq E$  such that at most one edge in  $M$  is incident on any vertex

#### 18.5.1.2 Bipartite Matching

**Problem 18.5.2 (Bipartite matching).**

**Input:** Given a bipartite graph  $G = (L \cup R, E)$ .

**Goal:** Find a matching of maximum cardinality

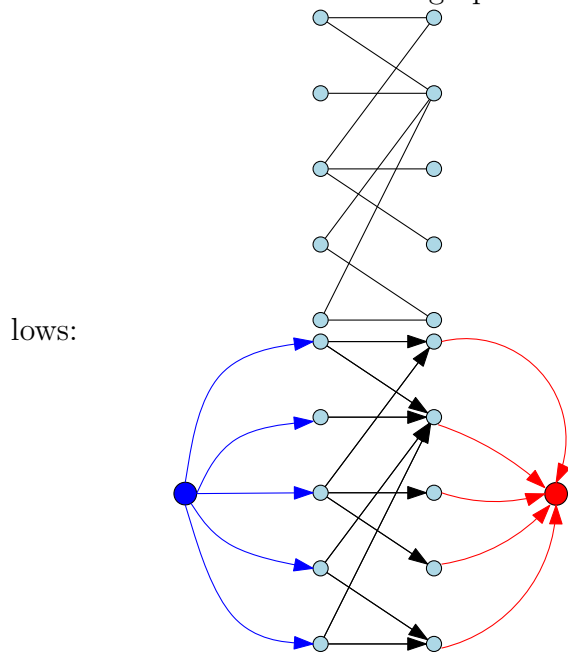


Maximum matching has 4 edges

## 18.5.2 Reduction of bipartite matching to max-flow

### 18.5.2.1 Reduction of bipartite matching to max-flow

Max-Flow Construction Given graph  $G = (L \cup R, E)$  create flow-network  $G' = (V', E')$  as follows:



- (A)  $V' = L \cup R \cup \{s, t\}$  where  $s$  and  $t$  are the new source and sink.
- (B) Direct all edges in  $E$  from  $L$  to  $R$ , and add edges from  $s$  to all vertices in  $L$  and from each vertex in  $R$  to  $t$ .
- (C) Capacity of every edge is 1.

### 18.5.2.2 Correctness: Matching to Flow

**Proposition 18.5.3.** *If  $G$  has a matching of size  $k$  then  $G'$  has a flow of value  $k$ .*

*Proof:* Let  $M$  be matching of size  $k$ . Let  $M = \{(u_1, v_1), \dots, (u_k, v_k)\}$ . Consider following flow  $f$  in  $G'$ :

- (A)  $f(s, u_i) = 1$  and  $f(v_i, t) = 1$  for  $1 \leq i \leq k$
- (B)  $f(u_i, v_i) = 1$  for  $1 \leq i \leq k$
- (C) for all other edges flow is zero.

Verify that  $f$  is a flow of value  $k$  (because  $M$  is a matching). ■

### 18.5.2.3 Correctness: Flow to Matching

**Proposition 18.5.4.** *If  $G'$  has a flow of value  $k$  then  $G$  has a matching of size  $k$ .*

*Proof:* Consider flow  $f$  of value  $k$ .

(A) Can assume  $f$  is integral. Thus each edge has flow 1 or 0.

(B) Consider the set  $M$  of edges from  $L$  to  $R$  that have flow 1.

(A)  $M$  has  $k$  edges because value of flow is equal to the number of non-zero flow edges crossing cut  $(L \cup \{s\}, R \cup \{t\})$

(B) Each vertex has at most one edge in  $M$  incident upon it. Why? ■

### 18.5.2.4 Correctness of Reduction

**Theorem 18.5.5.** *The maximum flow value in  $G'$  = maximum cardinality of matching in  $G$ .*

Consequence Thus, to find maximum cardinality matching in  $G$ , we construct  $G'$  and find the maximum flow in  $G'$ . Note that the matching itself (not just the value) can be found efficiently from the flow.

### 18.5.2.5 Running Time

For graph  $G$  with  $n$  vertices and  $m$  edges  $G'$  has  $O(n + m)$  edges, and  $O(n)$  vertices.

(A) Generic Ford-Fulkerson: Running time is  $O(mC) = O(nm)$  since  $C = n$ .

(B) Capacity scaling: Running time is  $O(m^2 \log C) = O(m^2 \log n)$ .

Better running time is known:  $O(m\sqrt{n})$ .

## 18.5.3 Perfect Matchings

### 18.5.3.1 Perfect Matchings

**Definition 18.5.6.** *A matching  $M$  is said to be **perfect** if every vertex has one edge in  $M$  incident upon it.*

### 18.5.3.2 Characterizing Perfect Matchings

Problem When does a bipartite graph have a perfect matching?

(A) Clearly  $|L| = |R|$

(B) Are there any necessary and sufficient conditions?

### 18.5.3.3 A Necessary Condition

**Lemma 18.5.7.** *If  $G = (L \cup R, E)$  has a perfect matching then for any  $X \subseteq L$ ,  $|N(X)| \geq |X|$ , where  $N(X)$  is the set of neighbors of vertices in  $X$ .*

*Proof:* Since  $G$  has a perfect matching, every vertex of  $X$  is matched to a different neighbor, and so  $|N(X)| \geq |X|$ . ■

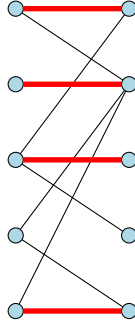


Figure 18.1: This graph does not have a perfect matching

### 18.5.3.4 Hall's Theorem

**Theorem 18.5.8 (Frobenius-Hall).** Let  $G = (L \cup R, E)$  be a bipartite graph with  $|L| = |R|$ .  $G$  has a perfect matching if and only if for every  $X \subseteq L$ ,  $|N(X)| \geq |X|$ .

One direction is the necessary condition.

For the other direction we will show the following:

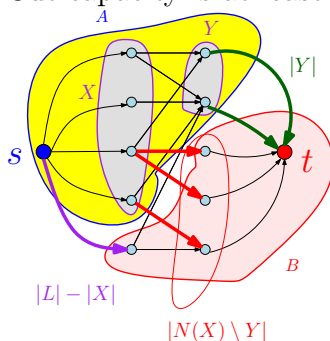
- (A) Create flow network  $G'$  from  $G$ .
- (B) If  $|N(X)| \geq |X|$  for all  $X$ , show that minimum  $s$ - $t$  cut in  $G'$  is of capacity  $n = |L| = |R|$ .
- (C) Implies that  $G$  has a perfect matching.

### 18.5.3.5 Proof of Sufficiency

Assume  $|N(X)| \geq |X|$  for any  $X \subseteq L$ . Then show that min  $s$ - $t$  cut in  $G'$  is of capacity at least  $n$ .

Let  $(A, B)$  be an arbitrary  $s$ - $t$  cut in  $G'$

- (A) Let  $X = A \cap L$  and  $Y = A \cap R$ .
- (B) Cut capacity is at least  $(|L| - |X|) + |Y| + |N(X) \setminus Y|$



Because there are...

- (A)  $|L| - |X|$  edges from  $s$  to  $L \cap B$ .
- (B)  $|Y|$  edges from  $Y$  to  $t$ .
- (C) there are at least  $|N(X) \setminus Y|$  edges from  $X$  to vertices on the right side that are not in  $Y$ .

## 18.5.4 Proof of Sufficiency

### 18.5.4.1 Continued...

- (A) By the above, cut capacity is at least

$$\alpha = (|L| - |X|) + |Y| + |N(X) \setminus Y|.$$

(B)  $|N(X) \setminus Y| \geq |N(X)| - |Y|$ .

(This holds for any two sets.)

(C) By assumption  $|N(X)| \geq |X|$  and hence

$$|N(X) \setminus Y| \geq |N(X)| - |Y| \geq |X| - |Y|.$$

(D) Cut capacity is therefore at least

$$\begin{aligned} \alpha &= (|L| - |X|) + |Y| + |N(X) \setminus Y| \\ &\geq |L| - |X| + |Y| + |X| - |Y| \geq |L| = n. \end{aligned}$$

(E) Any  $s$ - $t$  cut capacity is at least  $n \implies$  max flow at least  $n$  units  $\implies$  perfect matching.

**QED**

#### 18.5.4.2 Hall's Theorem: Generalization

**Theorem 18.5.9 (Frobenius-Hall).** *Let  $G = (L \cup R, E)$  be a bipartite graph with  $|L| \leq |R|$ .  $G$  has a matching that matches all nodes in  $L$  if and only if for every  $X \subseteq L$ ,  $|N(X)| \geq |X|$ .*

Proof is essentially the same as the previous one.

#### 18.5.4.3 Assigning jobs to people

(A)  $n$  jobs,  $n/2$  people

(B) For each job: a set of people who can do that job.

(C) Each person  $j$  has to do exactly two jobs.

(D) **Goal:** find an assignment of 2 jobs to each person, such that all jobs are assigned.

**Solution:** Build bipartite graph, compute maximum matching, remove it, compute another maximum matching. Both matchings together form a valid solution if it exists. This algorithm is

(A) Correct.

(B) Incorrect.

#### 18.5.4.4 Application: Assigning jobs to people

(A)  $n$  jobs or tasks

(B)  $m$  people

(C) for each job a set of people who can do that job

(D) for each person  $j$  a limit on number of jobs  $k_j$

(E) **Goal:** find an assignment of jobs to people so that all jobs are assigned and no person is overloaded

Reduce to max-flow similar to matching. Arises in many settings. Using *minimum-cost flows* can also handle the case when assigning a job  $i$  to person  $j$  costs  $c_{ij}$  and goal is assign all jobs but minimize cost of assignment.

#### 18.5.4.5 Reduction to Maximum Flow

- (A) Create directed graph  $G = (V, E)$  as follows
  - (A)  $V = \{s, t\} \cup L \cup R$ :  $L$  set of  $n$  jobs,  $R$  set of  $m$  people
  - (B) add edges  $(s, i)$  for each job  $i \in L$ , capacity 1
  - (C) add edges  $(j, t)$  for each person  $j \in R$ , capacity  $k_j$
  - (D) if job  $i$  can be done by person  $j$  add an edge  $(i, j)$ , capacity 1
- (B) Compute max  $s$ - $t$  flow. There is an assignment if and only if flow value is  $n$ .

#### 18.5.4.6 Matchings in General Graphs

Matchings in general graphs more complicated.

There is a polynomial time algorithm to compute a maximum matching in a general graph. Best known running time is  $O(m\sqrt{n})$ .

# Bibliography

Menger, K. (1927). Zur allgemeinen Kruventheorie. *Fund. Math.*, 10:96–115.