Breadth First Search, Dijkstra's Algorithm for Shortest Paths

Lecture 3

January 24, 2013

Part I

Breadth First Search

Breadth First Search (BFS)

Overview

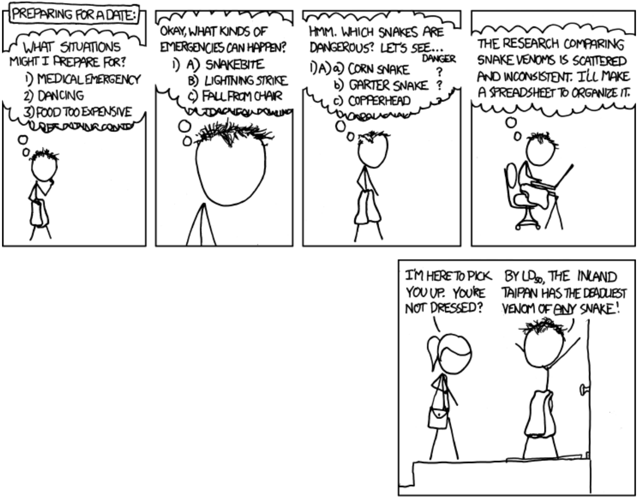
(A) **BFS** is obtained from **BasicSearch** by processing edges using a data structure called a **queue**.

(B) It processes the vertices in the graph in the order of their shortest distance from the vertex **s** (the start vertex).

As such...

1. **DFS** good for exploring graph structure
2. **BFS** good for exploring *distances*

xkcd take on DFS



I REALLY NEED TO STOP USING DEPTH-FIRST SEARCHES.

Queue Data Structure

Queues

A **queue** is a list of elements which supports the operations:

1. **enqueue**: Adds an element to the end of the list
2. **dequeue**: Removes an element from the front of the list

Elements are extracted in **first-in first-out (FIFO)** order, i.e., elements are picked in the order in which they were inserted.

BFS Algorithm

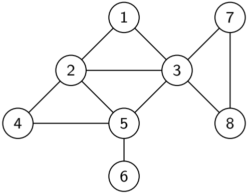
Given (undirected or directed) graph $G = (V, E)$ and node $s \in V$

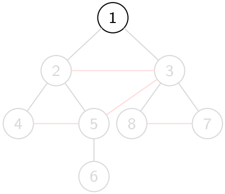
```
BFS(s)
    Mark all vertices as unvisited
    Initialize search tree T to be empty
    Mark vertex s as visited
    set Q to be the empty queue
    enq(s)
    while Q is nonempty do
        u = deq(Q)
        for each vertex v ∈ Adj(u)
            if v is not visited then
                add edge (u, v) to T
                Mark v as visited and enq(v)
```

Proposition

BFS(s) *runs in* $O(n + m)$ *time.*

BFS: An Example in Undirected Graphs





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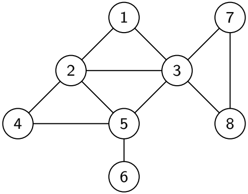
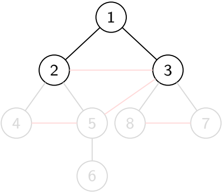
BFS tree is the set of black edges.

BFS: An Example in Undirected Graphs





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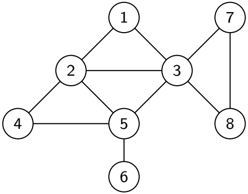

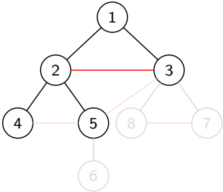

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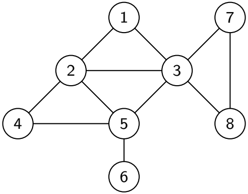

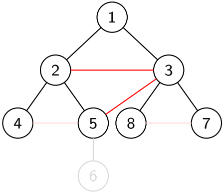

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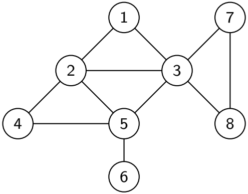

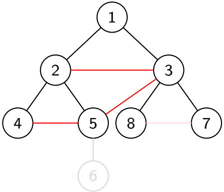

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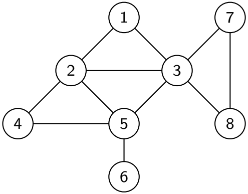

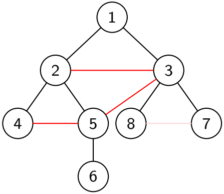

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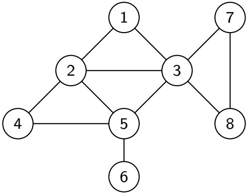

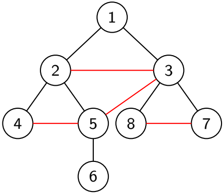

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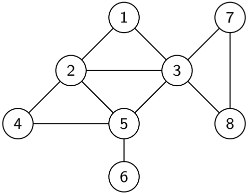

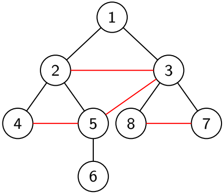

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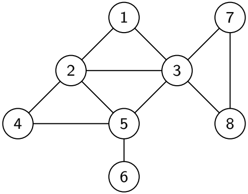

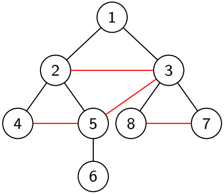

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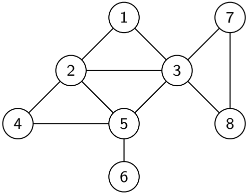

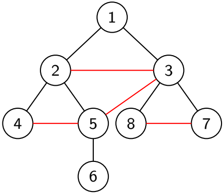

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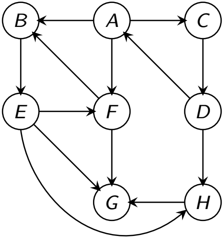
BFS: An Example in Directed Graphs



BFS with Distance

```
BFS(s)
    Mark all vertices as unvisited and for each v set dist(v) =
    Initialize search tree T to be empty
    Mark vertex s as visited and set dist(s) = 0
    set Q to be the empty queue
    enq(s)
    while Q is nonempty do
        u = deq(Q)
        for each vertex v ∈ Adj(u) do
            if v is not visited do
                add edge (u, v) to T
                Mark v as visited, enq(v)
                and set dist(v) = dist(u) + 1
```

Properties of BFS: Undirected Graphs

Proposition

The following properties hold upon termination of **BFS**$(\mathbf{s})$

(A) *The search tree contains exactly the set of vertices in the connected component of* $\mathbf{s}$.

(B) *If* $\mathrm{dist}(\mathbf{u}) < \mathrm{dist}(\mathbf{v})$ *then* $\mathbf{u}$ *is visited before* $\mathbf{v}$.

(C) *For every vertex* $\mathbf{u}$, $\mathrm{dist}(\mathbf{u})$ *is indeed the length of shortest path from* $\mathbf{s}$ *to* $\mathbf{u}$.

(D) *If* $\mathbf{u}, \mathbf{v}$ *are in connected component of* $\mathbf{s}$ *and* $\mathbf{e} = \{\mathbf{u}, \mathbf{v}\}$ *is an edge of* $\mathbf{G}$, *then either* $\mathbf{e}$ *is an edge in the search tree, or* $|\mathrm{dist}(\mathbf{u}) - \mathrm{dist}(\mathbf{v})| \leq \mathbf{1}$.

Proof.

Exercise. □

Properties of BFS: Directed Graphs

Proposition

The following properties hold upon termination of **BFS**$(\mathbf{s})$*:*

(A) *The search tree contains exactly the set of vertices reachable from* $\mathbf{s}$

(B) *If* $\mathrm{dist}(\mathbf{u}) < \mathrm{dist}(\mathbf{v})$ *then* $\mathbf{u}$ *is visited before* $\mathbf{v}$

(C) *For every vertex* $\mathbf{u}$, $\mathrm{dist}(\mathbf{u})$ *is indeed the length of shortest path from* $\mathbf{s}$ *to* $\mathbf{u}$

(D) *If* $\mathbf{u}$ *is reachable from* $\mathbf{s}$ *and* $\mathbf{e} = (\mathbf{u}, \mathbf{v})$ *is an edge of* $\mathbf{G}$, *then either* $\mathbf{e}$ *is an edge in the search tree, or* $\mathrm{dist}(\mathbf{v}) - \mathrm{dist}(\mathbf{u}) \leq 1$.
Not necessarily the case that $\mathrm{dist}(\mathbf{u}) - \mathrm{dist}(\mathbf{v}) \leq 1$.

Proof.

Exercise. □

BFS with Layers

```
BFSLayers(s):
    Mark all vertices as unvisited and initialize T to be empty
    Mark s as visited and set L_0 = {s}
    i = 0
    while L_i is not empty do
            initialize L_{i+1} to be an empty list
            for each u in L_i do
                for each edge (u, v) ∈ Adj(u) do
                if v is not visited
                        mark v as visited
                        add (u, v) to tree T
                        add v to L_{i+1}
            i = i + 1
```

Running time: $O(n + m)$

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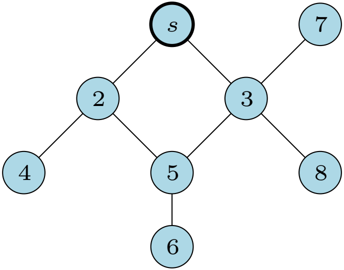
Example



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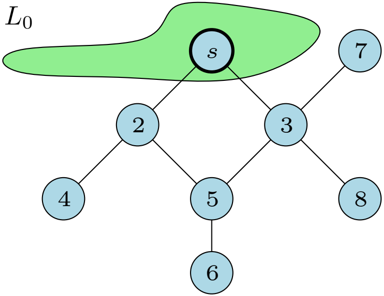

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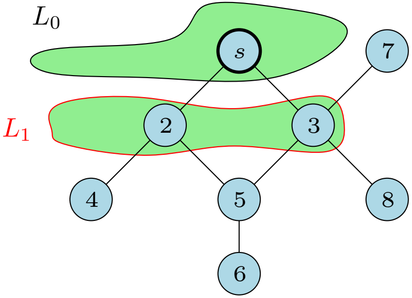

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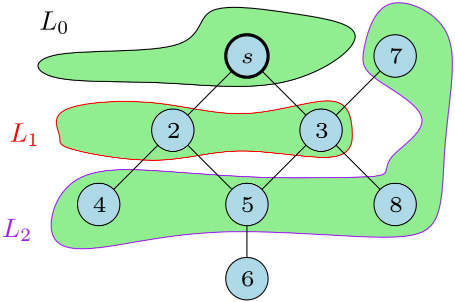

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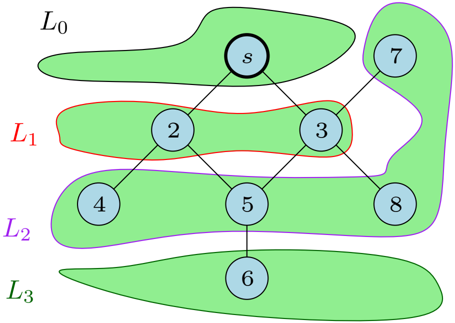


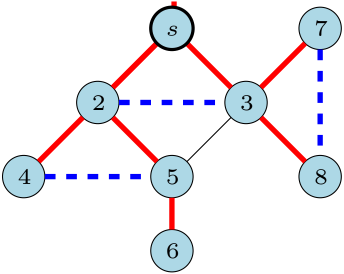
BFS with Layers: Properties

Proposition

The following properties hold on termination of **BFSLayers**($\mathbf{s}$).

1. **BFSLayers**($\mathbf{s}$) *outputs a* **BFS** *tree*
2. $\mathbf{L_i}$ *is the set of vertices at distance exactly* $\mathbf{i}$ *from* $\mathbf{s}$
3. *If* $\mathbf{G}$ *is undirected, each edge* $\mathbf{e} = \{\mathbf{u}, \mathbf{v}\}$ *is one of three types:*
   1. **tree** *edge between two* consecutive *layers*
   2. *non-tree* **forward/backward** *edge between two consecutive layers*
   3. *non-tree* **cross-edge** *with both* $\mathbf{u}, \mathbf{v}$ *in same layer*
   4. $\implies$ *Every edge in the graph is either between two vertices that are either (i) in the same layer, or (ii) in two consecutive layers.*

Example



BFS with Layers: Properties

For directed graphs

Proposition

The following properties hold on termination of **BFSLayers**(s)*, if* G *is directed.*

For each edge $e = (u, v)$ *is one of four types:*

1. *a* **tree** *edge between consecutive layers,* $u \in L_i, v \in L_{i+1}$ *for some* $i \geq 0$
2. *a non-tree* **forward** *edge between consecutive layers*
3. *a non-tree* **backward** *edge*
4. *a* **cross-edge** *with both* u, v *in same layer*

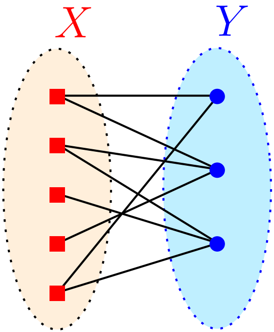
Part II

Bipartite Graphs and an application of BFS

Bipartite Graphs

Definition (Bipartite Graph)

Undirected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ is a **bipartite graph** if $\mathbf{V}$ can be partitioned into $\mathbf{X}$ and $\mathbf{Y}$ s.t. all edges in $\mathbf{E}$ are between $\mathbf{X}$ and $\mathbf{Y}$.



Bipartite Graph Characterization

Question

When is a graph bipartite?

Proposition

Every tree is a bipartite graph.

Proof.

Root tree $\mathbf{T}$ at some node $\mathbf{r}$. Let $\mathbf{L_i}$ be all nodes at level $\mathbf{i}$, that is, $\mathbf{L_i}$ is all nodes at distance $\mathbf{i}$ from root $\mathbf{r}$. Now define $\mathbf{X}$ to be all nodes at even levels and $\mathbf{Y}$ to be all nodes at odd level. Only edges in $\mathbf{T}$ are between levels. □

Proposition

An odd length cycle is not bipartite.

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Odd Cycles are not Bipartite

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An odd length cycle is not bipartite.

Proof.

Let $\mathbf{C} = \mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_{2k+1}}, \mathbf{u_1}$ be an odd cycle. Suppose $\mathbf{C}$ is a bipartite graph and let $\mathbf{X}, \mathbf{Y}$ be the partition. Without loss of generality $\mathbf{u_1} \in \mathbf{X}$. Implies $\mathbf{u_2} \in \mathbf{Y}$. Implies $\mathbf{u_3} \in \mathbf{X}$. Inductively, $\mathbf{u_i} \in \mathbf{X}$ if $\mathbf{i}$ is odd $\mathbf{u_i} \in \mathbf{Y}$ if $\mathbf{i}$ is even. But $\{\mathbf{u_1}, \mathbf{u_{2k+1}}\}$ is an edge and both belong to $\mathbf{X}$! □

Subgraphs

Definition

Given a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ a **subgraph** of $\mathbf{G}$ is another graph $\mathbf{H} = (\mathbf{V}', \mathbf{E}')$ where $\mathbf{V}' \subseteq \mathbf{V}$ and $\mathbf{E}' \subseteq \mathbf{E}$.

Proposition

If $\mathbf{G}$ is bipartite then any subgraph $\mathbf{H}$ of $\mathbf{G}$ is also bipartite.

Proposition

A graph $\mathbf{G}$ is not bipartite if $\mathbf{G}$ has an odd cycle $\mathbf{C}$ as a subgraph.

Proof.

If $\mathbf{G}$ is bipartite then since $\mathbf{C}$ is a subgraph, $\mathbf{C}$ is also bipartite (by above proposition). However, $\mathbf{C}$ is not bipartite! □

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Bipartite Graph Characterization

Theorem

A graph $\mathbf{G}$ *is bipartite if and only if it has no odd length cycle as subgraph.*

Proof.

Only If: $\mathbf{G}$ has an odd cycle implies $\mathbf{G}$ is not bipartite.

If: $\mathbf{G}$ has no odd length cycle. Assume without loss of generality that $\mathbf{G}$ is connected.

1. Pick $\mathbf{u}$ arbitrarily and do $\mathbf{BFS}(\mathbf{u})$
2. $\mathbf{X} = \cup_{\mathbf{i} \text{ is even}} \mathbf{L_i}$ and $\mathbf{Y} = \cup_{\mathbf{i} \text{ is odd}} \mathbf{L_i}$
3. **Claim:** $\mathbf{X}$ and $\mathbf{Y}$ is a valid partition if $\mathbf{G}$ has no odd length cycle.

□

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1. Pick $\mathbf{u}$ arbitrarily and do **BFS**($\mathbf{u}$)
2. $\mathbf{X} = \cup_{\mathbf{i} \text{ is even}} \mathbf{L_i}$ and $\mathbf{Y} = \cup_{\mathbf{i} \text{ is odd}} \mathbf{L_i}$
3. **Claim:** $\mathbf{X}$ and $\mathbf{Y}$ is a valid partition if $\mathbf{G}$ has no odd length cycle.

□

Proof of Claim

Claim

In **BFS**$(\mathbf{u})$ *if* $\mathbf{a}, \mathbf{b} \in \mathbf{L_i}$ *and* $(\mathbf{a}, \mathbf{b})$ *is an edge then there is an odd length cycle containing* $(\mathbf{a}, \mathbf{b})$.

Proof.

Let $\mathbf{v}$ be least common ancestor of $\mathbf{a}, \mathbf{b}$ in **BFS** tree $\mathbf{T}$.
$\mathbf{v}$ is in some level $\mathbf{j} < \mathbf{i}$ (could be $\mathbf{u}$ itself).
Path from $\mathbf{v} \rightsquigarrow \mathbf{a}$ in $\mathbf{T}$ is of length $\mathbf{j} - \mathbf{i}$.
Path from $\mathbf{v} \rightsquigarrow \mathbf{b}$ in $\mathbf{T}$ is of length $\mathbf{j} - \mathbf{i}$.
These two paths plus $(\mathbf{a}, \mathbf{b})$ forms an odd cycle of length $2(\mathbf{j} - \mathbf{i}) + 1$. □

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Proof of Claim: Figure

Another tidbit

Corollary

There is an $\mathbf{O(n + m)}$ *time algorithm to check if* $\mathbf{G}$ *is bipartite and output an odd cycle if it is not.*

Part III

Shortest Paths and Dijkstra's Algorithm

Shortest Path Problems

Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

1. Given nodes s, t find shortest path from s to t.
2. Given node s find shortest path from s to all other nodes.
3. Find shortest paths for all pairs of nodes.

Many applications!

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Single-Source Shortest Paths:

Non-Negative Edge Lengths

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2. Undirected graph problem can be reduced to directed graph problem - how?
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   3. Exercise: show reduction works

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Single-Source Shortest Paths via BFS

Special case: All edge lengths are 1.

1. Run **BFS**(s) to get shortest path distances from s to all other nodes.
2. $O(m + n)$ time algorithm.

Special case: Suppose $\ell(e)$ is an integer for all e?
Can we use **BFS**? Reduce to unit edge-length problem by placing $\ell(e) - 1$ dummy nodes on e

Let $L = \max_e \ell(e)$. New graph has $O(mL)$ edges and $O(mL + n)$ nodes. **BFS** takes $O(mL + n)$ time. Not efficient if L is large.

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Towards an algorithm

Why does **BFS** work?

BFS(s) explores nodes in increasing distance from s

Lemma

Let G be a directed graph with non-negative edge lengths. Let $\mathrm{dist}(s, v)$ denote the shortest path length from s to v. If $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k$ is a shortest path from s to v_k then for $1 \le i < k$:

1. *$s = v_0 \to v_1 \to v_2 \to \ldots \to v_i$ is a shortest path from s to v_i*
2. *$\mathrm{dist}(s, v_i) \le \mathrm{dist}(s, v_k)$.*

Proof.

Suppose not. Then for some $i < k$ there is a path P' from s to v_i of length strictly less than that of $s = v_0 \to v_1 \to \ldots \to v_i$. Then P' concatenated with $v_i \to v_{i+1} \ldots \to v_k$ contains a strictly shorter

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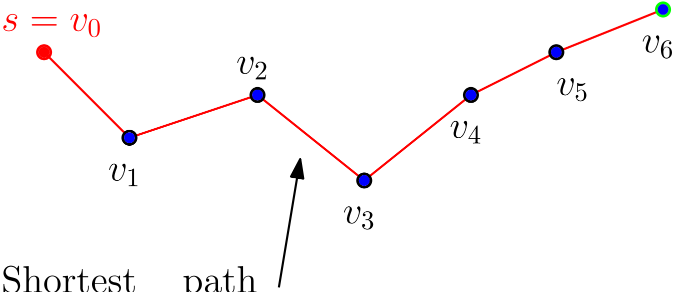
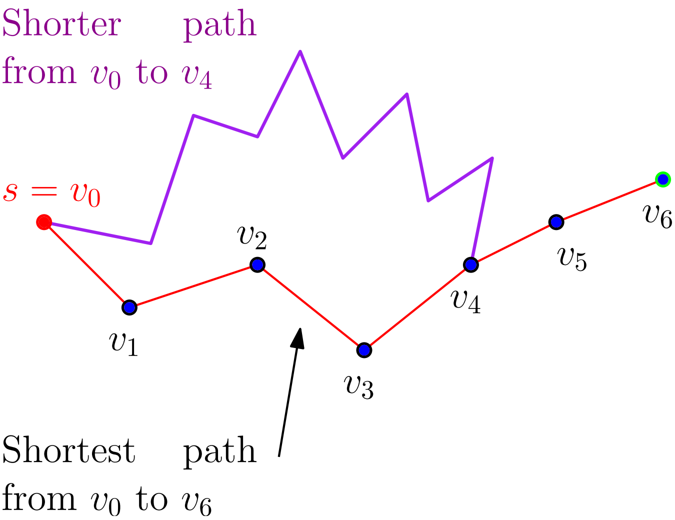
A proof by picture



Shortest path from v_0 to v_6

A proof by picture



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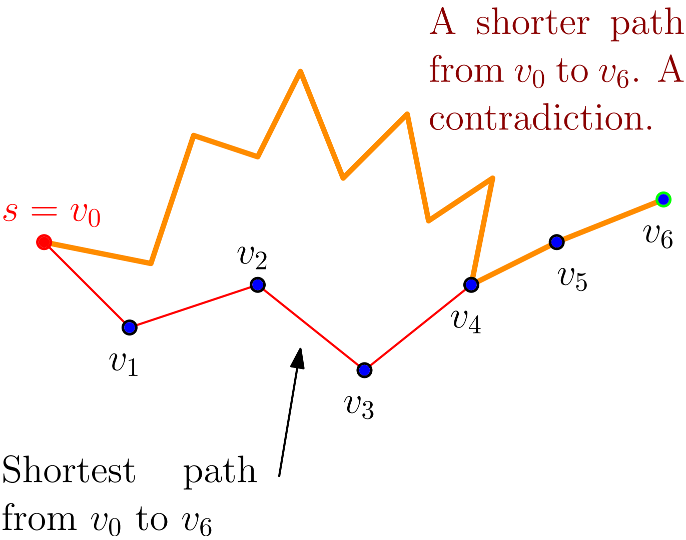

A Basic Strategy

Explore vertices in increasing order of distance from s:
(For simplicity assume that nodes are at different distances from s and that no edge has zero length)

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Initialize for each node v, dist(s,v) = ∞
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for i = 1 to |V| do
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How can we implement the step in the for loop?

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1. S contains the $i-1$ closest nodes to s
2. Want to find the ith closest node from $V-S$.

What do we know about the ith closest node?

Claim

Let P be a shortest path from s to v where v is the ith closest node. Then, all intermediate nodes in P belong to S.

Proof.

If P had an intermediate node u not in S then u will be closer to s than v. Implies v is not the ith closest node to s - recall that S already has the $i-1$ closest nodes. □

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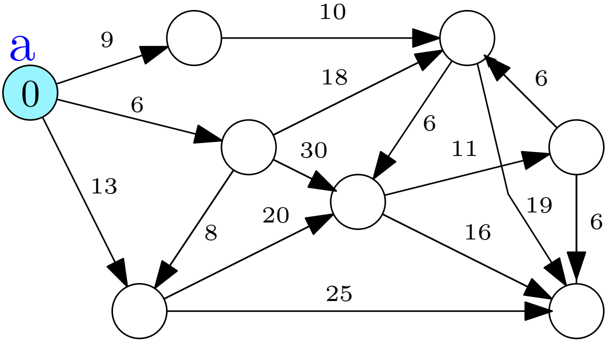
Finding the ith closest node repeatedly

An example



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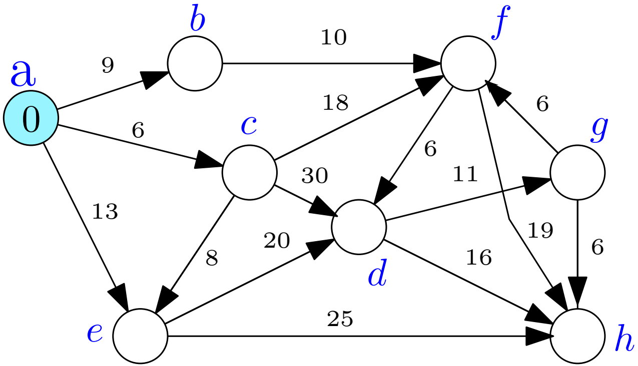

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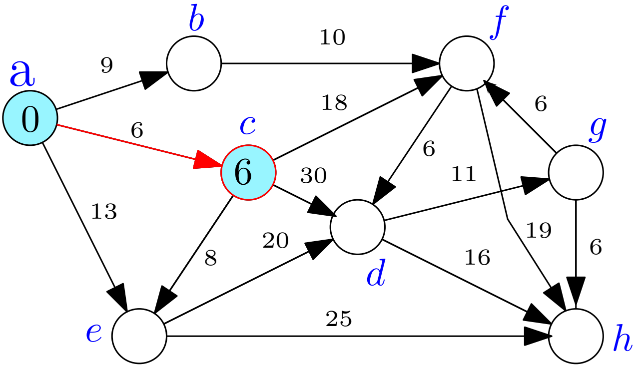

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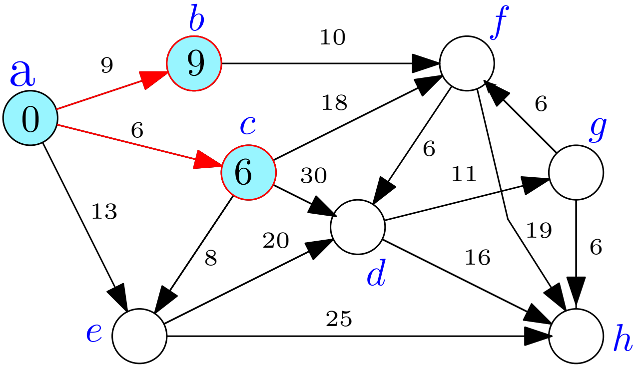

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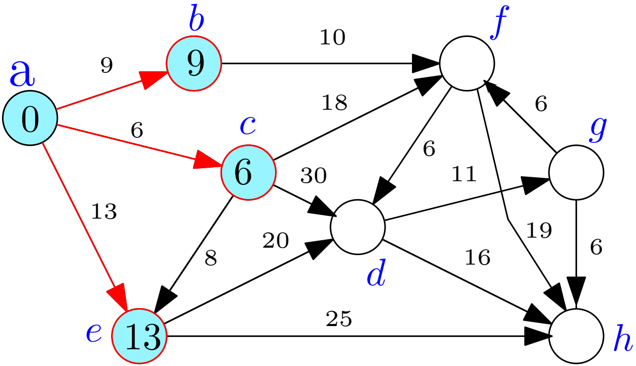

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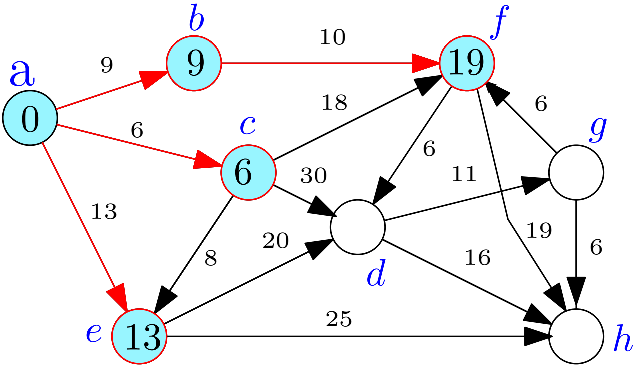

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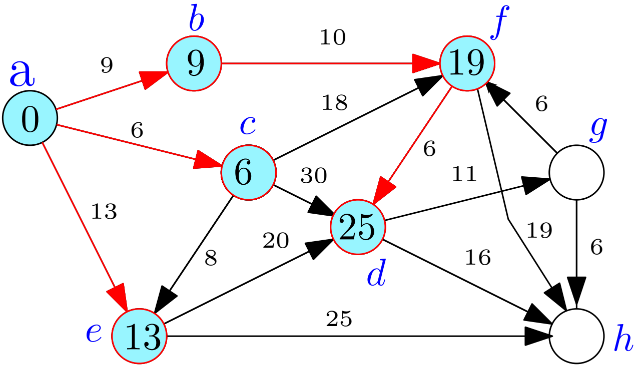

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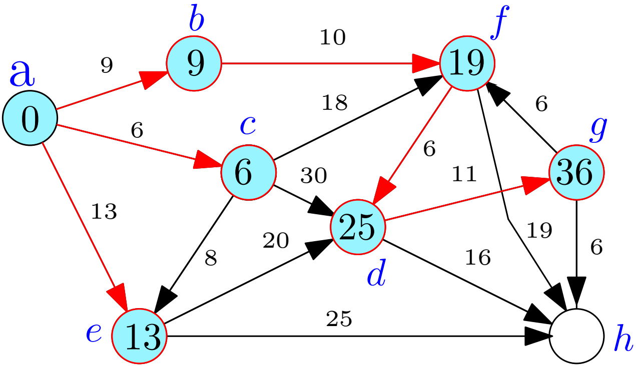

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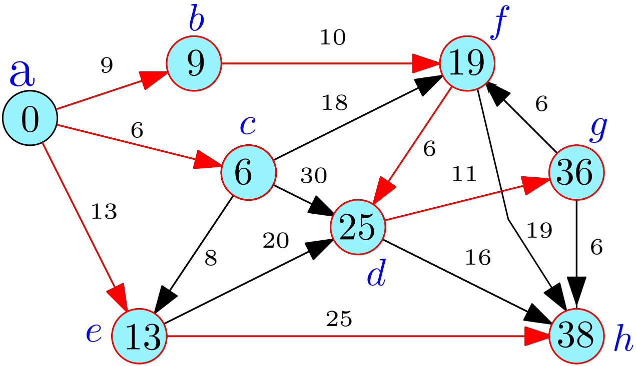

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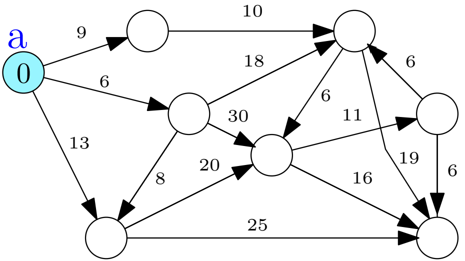

Corollary

The **i***th closest node is adjacent to* **S**.

Finding the ith closest node

1. $\mathbf{S}$ contains the $\mathbf{i-1}$ closest nodes to $\mathbf{s}$
2. Want to find the $\mathbf{i}$th closest node from $\mathbf{V-S}$.

1. For each $\mathbf{u \in V - S}$ let $\mathbf{P(s, u, S)}$ be a shortest path from $\mathbf{s}$ to $\mathbf{u}$ using only nodes in $\mathbf{S}$ as intermediate vertices.
2. Let $\mathbf{d'(s, u)}$ be the length of $\mathbf{P(s, u, S)}$

Observations: for each $\mathbf{u \in V - S}$,

1. $\mathrm{dist}(\mathbf{s}, \mathbf{u}) \leq \mathbf{d'(s, u)}$ since we are constraining the paths
2. $\mathbf{d'(s, u)} = \mathbf{min_{a \in S}}(\mathrm{dist}(\mathbf{s}, \mathbf{a}) + \ell(\mathbf{a}, \mathbf{u}))$ - Why?

Lemma

If $\mathbf{v}$ *is the* $\mathbf{i}$*th closest node to* $\mathbf{s}$*, then* $\mathbf{d'(s, v)} = \mathrm{dist}(\mathbf{s}, \mathbf{v})$.

Finding the ith closest node

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Finding the ith closest node

Lemma

Given:

1. $\mathbf{S}$: *Set of* $\mathbf{i-1}$ *closest nodes to* $\mathbf{s}$.
2. $\mathbf{d'(s,u) = min_{x\in S}(\mathrm{dist}(s,x) + \ell(x,u))}$

If $\mathbf{v}$ *is an* $\mathbf{i}$*th closest node to* $\mathbf{s}$*, then* $\mathbf{d'(s,v) = \mathrm{dist}(s,v)}$.

Proof.

Let $\mathbf{v}$ be the $\mathbf{i}$th closest node to $\mathbf{s}$. Then there is a shortest path $\mathbf{P}$ from $\mathbf{s}$ to $\mathbf{v}$ that contains only nodes in $\mathbf{S}$ as intermediate nodes (see previous claim). Therefore $\mathbf{d'(s,v) = \mathrm{dist}(s,v)}$. □

Finding the ith closest node

Lemma

If $\mathbf{v}$ *is an* i*th closest node to* $\mathbf{s}$*, then* $\mathbf{d'(s, v)} = \mathrm{dist}(\mathbf{s}, \mathbf{v})$*.*

Corollary

The i*th closest node to* $\mathbf{s}$ *is the node* $\mathbf{v} \in \mathbf{V} - \mathbf{S}$ *such that* $\mathbf{d'(s, v)} = \min_{\mathbf{u} \in \mathbf{V} - \mathbf{S}} \mathbf{d'(s, u)}$*.*

Proof.

For every node $\mathbf{u} \in \mathbf{V} - \mathbf{S}$, $\mathrm{dist}(\mathbf{s}, \mathbf{u}) \leq \mathbf{d'(s, u)}$ and for the ith closest node $\mathbf{v}$, $\mathrm{dist}(\mathbf{s}, \mathbf{v}) = \mathbf{d'(s, v)}$. Moreover, $\mathrm{dist}(\mathbf{s}, \mathbf{u}) \geq \mathrm{dist}(\mathbf{s}, \mathbf{v})$ for each $\mathbf{u} \in \mathbf{V} - \mathbf{S}$. □

Algorithm

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Initialize for each node v:  dist(s, v) = ∞
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Correctness: By induction on i using previous lemmas.

Running time: $O(n \cdot (n + m))$ time.

1. n outer iterations. In each iteration, $d'(s, u)$ for each u by scanning all edges out of nodes in S; $O(m + n)$ time/iteration.

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Correctness: By induction on i using previous lemmas.

Running time: $O(n \cdot (n + m))$ time.

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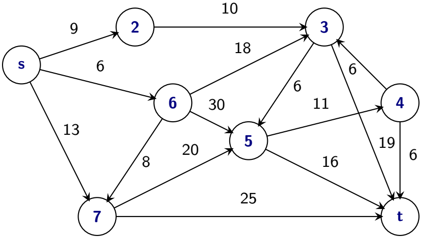

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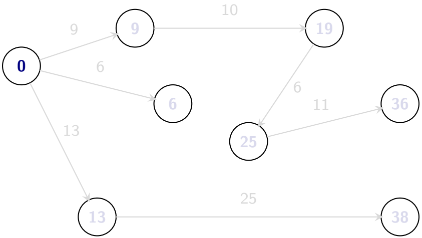

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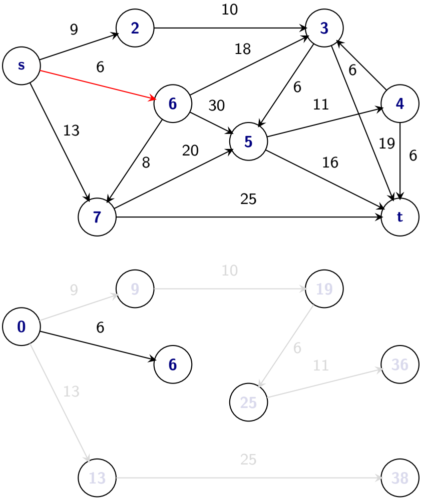

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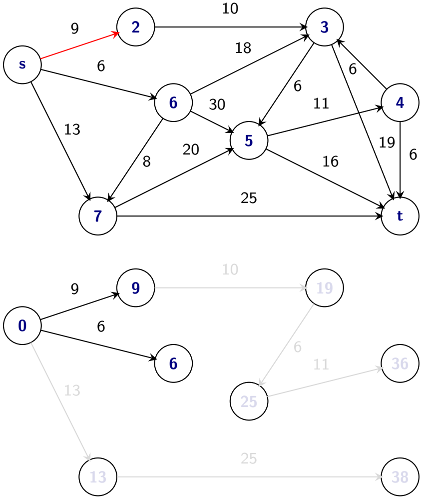

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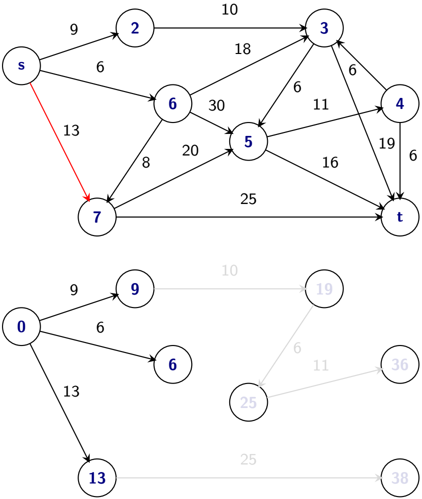

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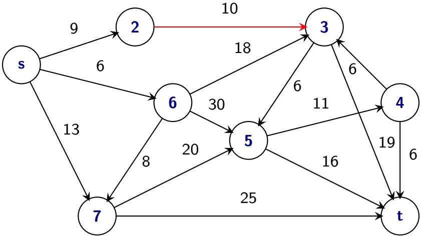
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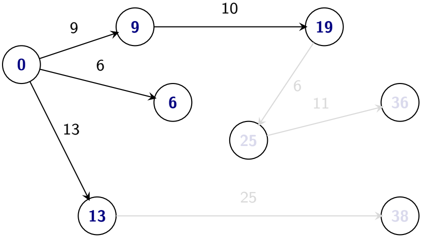
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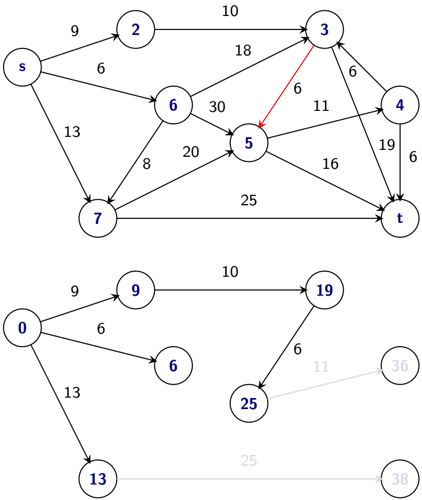

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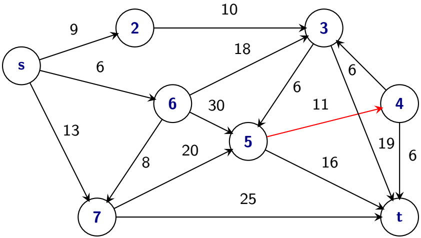

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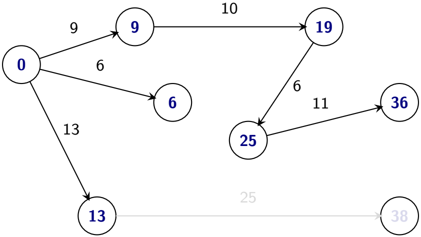

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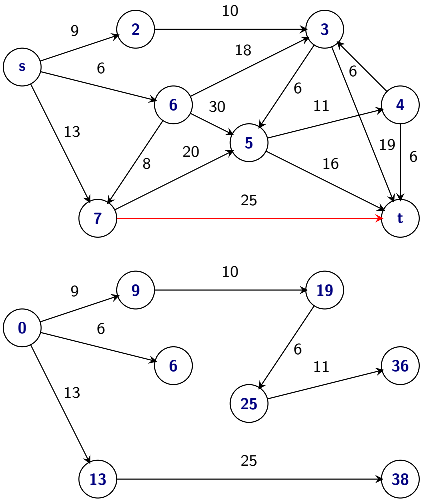
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Improved Algorithm

1. Main work is to compute the $\mathbf{d'(s,u)}$ values in each iteration
2. $\mathbf{d'(s,u)}$ changes from iteration $\mathbf{i}$ to $\mathbf{i+1}$ only because of the node $\mathbf{v}$ that is added to $\mathbf{S}$ in iteration $\mathbf{i}$.

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Running time: $\mathbf{O(m+n^2)}$ time.

1. $\mathbf{n}$ outer iterations and in each iteration following steps
2. updating $\mathbf{d'(s,u)}$ after $\mathbf{v}$ added takes $\mathbf{O(deg(v))}$ time so total

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3. Finding v from $d'(s, u)$ values is $O(n)$ time

Dijkstra's Algorithm

1. eliminate $\mathbf{d}'(\mathbf{s},\mathbf{u})$ and let $\mathrm{dist}(\mathbf{s},\mathbf{u})$ maintain it
2. update **dist** values after adding **v** by scanning edges out of **v**

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Priority Queues to maintain **dist** values for faster running time

1. Using heaps and standard priority queues: $O((m+n)\log n)$
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Priority Queues

Data structure to store a set $\mathbf{S}$ of $\mathbf{n}$ elements where each element $\mathbf{v} \in \mathbf{S}$ has an associated real/integer key $\mathbf{k(v)}$ such that the following operations:

1. **makePQ**: create an empty queue.
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All operations can be performed in $\mathbf{O(\log n)}$ time.
decreaseKey is implemented via **delete** and **insert**.

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Dijkstra's Algorithm using Priority Queues

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for each node u ≠ s do
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Priority Queue operations:

1. $O(n)$ **insert** operations
2. $O(n)$ **extractMin** operations
3. $O(m)$ **decreaseKey** operations

Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

1. All operations can be done in $O(\log n)$ time

Dijkstra's algorithm can be implemented in $O((n + m) \log n)$ time.

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Priority Queues: Fibonacci Heaps/Relaxed Heaps

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3. Relaxed Heaps: **decreaseKey** in $\mathbf{O(1)}$ worst case time but at the expense of **meld** (not necessary for Dijkstra's algorithm)

1. Dijkstra's algorithm can be implemented in $\mathbf{O(n \log n + m)}$ time. If $\mathbf{m = \Omega(n \log n)}$, running time is linear in input size.
2. Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)

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Shortest Path Tree

Dijkstra's algorithm finds the shortest path distances from s to **V**.

Question: How do we find the paths themselves?

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Shortest Path Tree

Lemma

The edge set $(u, \mathrm{prev}(u))$ *is the* reverse *of a shortest path tree rooted at* s. *For each* u, *the reverse of the path from* u *to* s *in the tree is a shortest path from* s *to* u.

Proof Sketch.

1. The edge set $\{(u, \mathrm{prev}(u)) \mid u \in V\}$ induces a directed in-tree rooted at s (Why?)
2. Use induction on $|S|$ to argue that the tree is a shortest path tree for nodes in V.

□

Shortest paths to s

Dijkstra's algorithm gives shortest paths from **s** to all nodes in **V**.
How do we find shortest paths from all of **V** to **s**?

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