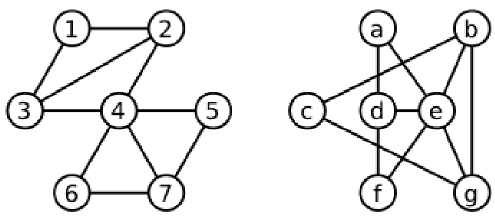
CS 473: Fundamental Algorithms, Spring 2013

Discussion 13

April 16, 2013

13.1. Graph Isomorphism.

Two graphs are said to be ***isomorphic*** if one can be transformed into the other by relabeling the vertices. For example, the graphs shown below are isomorphic; the left graph can be transformed into the right graph by the relabeling $(1, 2, 3, 4, 5, 6, 7) \implies (c, g, b, e, a, f, d)$.



Two isomorphic graphs.

Consider the following related decision problems:

- ⋆ Graph Isomorphism: Given two graphs G and H, determine whether G and H are isomorphic.
- ⋆ Even Graph Isomorphism: Given two graphs G and H, such that every vertex in G and H has even degree, determine whether G and H are isomorphic.
- ⋆ Subgraph Isomorphism: Given two graphs G and H, determine whether G is isomorphic to a subgraph of H.

(A) Describe a polynomial-time reduction from Graph Isomorphism to Even Graph Isomorphism.

(B) Describe a polynomial-time reduction from Graph Isomorphism to Subgraph Isomorphism.

(C) Prove that Subgraph Isomorphism is NP-Complete by reducing from Clique. Can you also reduce from Hamiltonian Cycle?

13.2. Self-Reductions.

In each case below, assume that you are given a black box which can answer the decision version of the indicated problem. Use a polynomial number of calls to the black box to construct the desired set.

(A) Subset sum: Given a multiset (elements can appear more than once) $X = x_1, \ldots, x_k$ of positive integers, and a positive integer S, does there exist a subset of X with sum exactly S?

(B) k-Color: Given a graph G, is there a proper k-coloring? In other words, can we assign one of the k colors to each node such that no node is adjacent to a node of the same color?

The same question for all the problems mentioned in (13.6).

13.3. Dominating Set.

Prove that the Dominating Set is NP-Complete via a reduction from Set Cover. In a graph G, a ***dominating set*** is a set $X \subseteq \mathsf{V}(\mathsf{G})$, such that each vertex of G is either in X, or adjacent to a vertex of X in the graph G.

Dominating Set

Instance: A graph G, and a constant $k > 0$.
Question: Is there a set $X \subseteq \mathsf{V}(\mathsf{G})$ of at most k vertices that is a dominating set?

13.4. Zero-Length Cycle.

Let $G = (V, E)$ be a directed graph that has weights on its edges; $w(e)$ represents the weight of edge e and it can be positive or negative. Given G the Zero-Length-Cycle problem is to check if G has a (simple) cycle C such that the sum of the weights on the edges in C is exactly equal to 0. Show that this problem is NP-Complete.

13.5. Partition.

The Partition problem is the following. Given n non-negative integers $a_1, a_2, \ldots, a_n$ is there a partition of the numbers into two sets such that the sum of the numbers in each set is exactly equal to $(\sum_i a_i)/2$. Show that this problem is NP-Complete.

13.6. Certificates of positive answer.

This problem was in the homeworks, so it is not for the discussion section this semester.

For each of the following problems, describe what is the certificate that testifies that the answer to the given instance is positive, and how do you verify that the certificate is indeed correct.

(A) Min Cut.
Input: A flow network G and subset C of the edges.
Question: Is the set C is a minimum cut in G?

(B) Max Cut.
Input: An undirected graph G, and an integer k.
Question: Does the graph G contains an undirected cut with at least k edges in it?

(C) Independent Set.
Input: $\langle G, k \rangle$.
Question: Does G contains an independent set of size k?

(D) Clique.
Input: $\langle G, k \rangle$.
Question: Does G contains a clique of size k?

(E) Hamiltonian Cycle.
Input: A directed graph G.
Question: Does G contains a Hamiltonian cycle?

(F) SAT.
Input: A CNF formula F.
Question: Is there a satisfying assignment for F?

(G) Set Cover.
Input: A ground set $U = \{1, \ldots, m\}$, a family of subsets $\mathcal{F} = \{F_1, \ldots, F_n\}$, and a number k. Here $F_i \subseteq U$, for all i.
Question: Are there k subsets $F_{i_1}, \ldots, F_{i_k} \in \mathcal{F}$ such that $\bigcup_{j=1}^{k} F_{i_j} = U$.

(H) Partition.
Input: A set $X = \{x_1, \ldots, x_n\}$ of n positive numbers.
Question: Is there a subset $S \subseteq X$, such that $\sum_{x \in S} x = \sum_{x \in X \setminus S} x$?

(I) Graph Isomorphism.
Input: Two graphs $G = (V, E)$ and $G' = (V', E')$.
Question: Is G isomorphic to G'?
That is, is there a bijection $f : V \to V'$ such that for all $u, v \in V$, we have that $uv \in E$ if and only if $f(u)f(v) \in E'$.

All the problems mentioned above, except for Min Cut (which is solvable in polynomial time) and Graph Isomorphism are NP-Complete. For Graph Isomorphism it is currently open if it is NP-Complete (note, that Subgraph Isomorphism is NP-Complete).