CS 473: Fundamental Algorithms, Spring 2013

Discussion 12

April 9, 2013

12.1. Building 3CNF formulas.

CNF formula (***conjunctive normal form***) is a boolean formula that is the 'and' of clauses, where every clause is the 'or' of literals, where every literal is either a variable or its negation. For example, a CNF formula is

$$\left(\overline{x_1} \vee x_2\right) \wedge \left(\overline{x_2} \vee x_3 \vee x_4 \vee \overline{x_5}\right) \wedge \left(\overline{x_2} \vee \overline{x_3} \vee x_4 \vee \overline{x_5}\right).$$

A formula is 3CNF if every clause contains exactly 3 literals that are of three distinct variables. An example of a 3CNF formula:

$$\left(\overline{x_1} \vee x_2 \vee \overline{x_5}\right) \wedge \left(\overline{x_2} \vee x_4 \vee \overline{x_5}\right) \wedge \left(\overline{x_1} \vee \vee x_4 \vee \overline{x_5}\right) \wedge \left(\overline{x_3} \vee x_4 \vee \overline{x_5}\right).$$

(A) Consider the following boolean function f and g defined by a truth table. Generate a 3CNF formulas that computes these two functions.

x	y	z	$f(x,y,z)$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

(i)

x	y	z	$g(x,y,z)$
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

(ii)

(B) Given an arbitrary boolean formula $f(x,y,z)$, describe how to convert it into an equivalent 3CNF formula.

(C) Argue that any boolean formula with n variables can be converted into a n-CNF formula (i.e., CNF formula where every clause has at most n variables).

12.2. From Set Cover to Monotone SAT.

Consider an instance I of a CNF formula specified by clauses $C_1, C_2, \ldots, C_k$ over a set of boolean variables $x_1, x_2, \ldots, x_n$. We say that I is ***monotone*** if each term in each clause consists of a nonnegated variable i.e. each term is equal to x_i, for some i, rather than $\overline{x_i}$ (i.e., no negations are allowed). They could be easily satisfied by setting each variable to 1. For example, suppose we have three clauses $(x_1 \vee x_2), (x_1 \vee x_3), (x_3 \vee x_2)$. These could be satisfied by setting all three variables to 1, or by setting x_1 and x_2 to 1 and x_3 to 0.

Given a monotone instance of CNF formula, together with a number k, the problem Monotone Satisfiability asks whether there is a satisfying assignment for the instance in which at most k variables are set to 1.

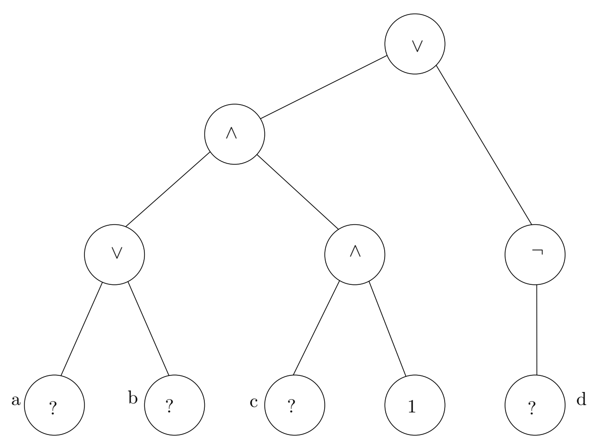
The Set Cover problem asks, given a collection $\mathcal{F}$ of subsets $S_1, S_2, \ldots, S_m$ of a ground set $U = \{1, \ldots, n\}$, what is the minimum number of sets of $\mathcal{F}$ whose union is U?

(A) Given a decision instance of Set Cover (i.e., given S, $\mathcal{F}$, and a k – is there a cover of U by k subsets?), show a Karp reduction to Monotone Satisfiability.

(B) Show how to solve the optimization version of Set Cover (i.e., you are given $U, \mathcal{F}$, and you have to compute the minimum number of sets of $\mathcal{F}$ that cover the ground set) by an algorithm performing a polynomial number of calls to a solver of Monotone Satisfiability.

12.3. From Circuit-SAT to SAT.

Convert the following Circuit-SAT instance into a SAT formula such that the resulting formula is satisfiable if and only if the curcuit sat instance is satisfiable. Use x_a, x_b, x_c, x_d as the variable names for the four unknowns shown in the figure. You may need additional variables.



12.4. Reducing from 3-coloring to SAT.

SAT is a decision problem that asks whether a given boolean formula in conjunctive normal form (CNF) has an assignment that makes the formula true. The 3-Coloring problem is a decision problem that asks given an undirected graph G, can its vertices be colored with three colors, so that every edge touches vertices with two different colors? Give a polynomial time reduction from 3-coloring to 3SAT.

Comment: We will show an intricate reduction in lecture from 3SAT to 3-coloring which shows that the latter problem is hard.