CS 473: Fundamental Algorithms, Spring 2011

HW 9

Question 1 is due by **Sunday, 23:59:59, April 10**

Questions 2-4 are due by **Monday, 23:59:59, April 18**

This homework contains four problems. **Read the instructions for submitting homework on the course webpage**.

Collaboration Policy: For this homework, Problems 2–4 can be worked in groups of up to three students.

Problem 1 should be answered in Compass as part of the assessment HW9-Online and should be done individually.

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1. HW9-Online. (25 pts.)

2. Transitivity. (25 pts.)

   Given a (directed) graph $G = (V, E)$, and a natural number k, we can define a relation $\xrightarrow{G,k}$ on pairs of vertices of G as follows. If $x, y \in V$, we say that $x \xrightarrow{G,k} y$ if there exist k mutually edge-disjoint paths from x to y in G.

   Is it true that for every G and every $k \geq 0$, the relation $\xrightarrow{G,k}$ is transitive? That is, is it always the case that if $x \xrightarrow{G,k} y$ and $y \xrightarrow{G,k} z$, then we have $x \xrightarrow{G,k} z$? Give a proof or a counterexample.

3. Census Rounding.
   (25 pts.)

   You are consulting for an environmental statistics firm. They collect statistics and publish the collected data in a book. The statistics are about populations of different regions in the world and are recorded in multiples of one million. Examples of such statistics would look like the following table.

Country	A	B	C	Total
grown-up men	11.998	9.083	2.919	24.000
grown-up women	12.983	10.872	3.145	27.000
children	1.019	2.045	0.936	4.000
total	26.000	22.000	7.000	55.000

   We will assume here for simplicity that our data is such that all row and column sums are integers. The Census Rounding Problem is to round all data to integers without changing any row or column sum. Each fractional number can be rounded either up or down. For example, a good rounding for our table data would be as follows.

Country	A	B	C	Total
grown-up men	11.000	10.000	3.000	24.000
grown-up women	13.000	10.000	4.000	27.000
children	2.000	2.000	0.000	4.000
total	26.000	22.000	7.000	55.000

(a) (5 pts.) Consider first the special case when all data are between 0 and 1. So you have a matrix of fractional numbers between 0 and 1, and your problem is to round each fraction that is between 0 and 1 to either 0 or 1 without changing the row or column sums. Use a flow computation to check if the desired rounding is possible.

(b) (5 pts.) Consider the Census Rounding Problem as defined above, where row and column sums are integers, and you want to round each fractional number α to either $\lfloor \alpha \rfloor$ or $\lceil \alpha \rceil$. Use a flow computation to check if the desired rounding is possible.

(c) (10 pts.) Prove that the rounding we are looking for in (a) and (b) always exists.

4. Minimum Flow (25 pts.)

Give a polynomial-time algorithm for the following minimization analogue of the Maximum-Flow Problem. You are given a directed graph $G = (V, E)$, with a source $s \in V$ and sink $t \in V$, and numbers (capacities) $\ell(v, w)$ for each edge $(v, w) \in E$. We define a flow f, and the value of a flow, as usual, requiring that all nodes except s and t satisfy flow conservation. However, the given numbers are lower bounds on edge flow – that is, they require that $f(v, w) \geq \ell(v, w)$ for every edge $(v, w) \in E$, and there is no upper bound on flow values on edges.

(a) Give a polynomial-time algorithm that finds a feasible flow of minimum possible values.

(b) Prove an analogue of the Max-Flow Min-Cut Theorem for this problem (i.e., does min-flow = max-cut?).