More NP-Complete Problems

Lecture 23

April 21, 2011

Recap

NP: languages that have polynomial time certifiers/verifiers

A language **L** is **NP-Complete** iff

- **L** is in **NP**
- for every **L′** in **NP**, $\mathbf{L'} \leq_{\mathbf{P}} \mathbf{L}$

L is **NP-Hard** if for every **L′** in NP, $\mathbf{L'} \leq_{\mathbf{P}} \mathbf{L}$.

Theorem (Cook-Levin)

Circuit-SAT *and* ***SAT*** *are* ***NP-Complete****.*

Recap contd

Theorem (Cook-Levin)

Circuit-SAT and SAT are **NP-Complete**.

Establish **NP-Complete**ness via reductions:

- SAT $\leq_P$ 3-SAT and hence 3-SAT is **NP**-complete
- 3-SAT $\leq_P$ Independent Set (which is in **NP**) and hence Independent Set is **NP-Complete**
- Vertex Cover is **NP-Complete**
- Clique is **NP-Complete**
- Set Cover is **NP-Complete**

Today

Prove

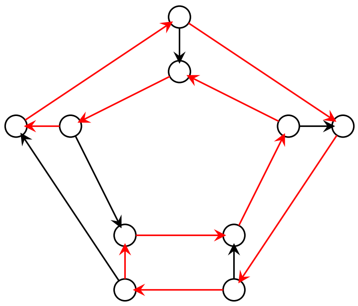
- Hamiltonian Cycle Problem is **NP-Complete**
- 3-Coloring is **NP-Complete**

Directed Hamiltonian Cycle

Input Given a directed graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with $\mathbf{n}$ vertices

Goal Does $\mathbf{G}$ have a Hamiltonian cycle?

- A Hamiltonian cycle is a cycle in the graph that visits every vertex in $\mathbf{G}$ exactly once



Directed Hamiltonian Cycle is NP-complete

- Directed Hamiltonian Cycle is in **NP**
  - Certificate: Sequence of vertices
  - Certifier: Check if every vertex (except the first) appears exactly once, and that consecutive vertices are connected by a directed edge
- Hardness: We will show **3-SAT $\leq_P$ Directed Hamiltonian Cycle**

Reduction

Given 3-SAT formula φ create a graph G_φ such that

- G_φ has a Hamiltonian cycle if and only if φ is satisfiable
- G_φ should be constructible from φ by a polynomial time algorithm $\mathcal{A}$

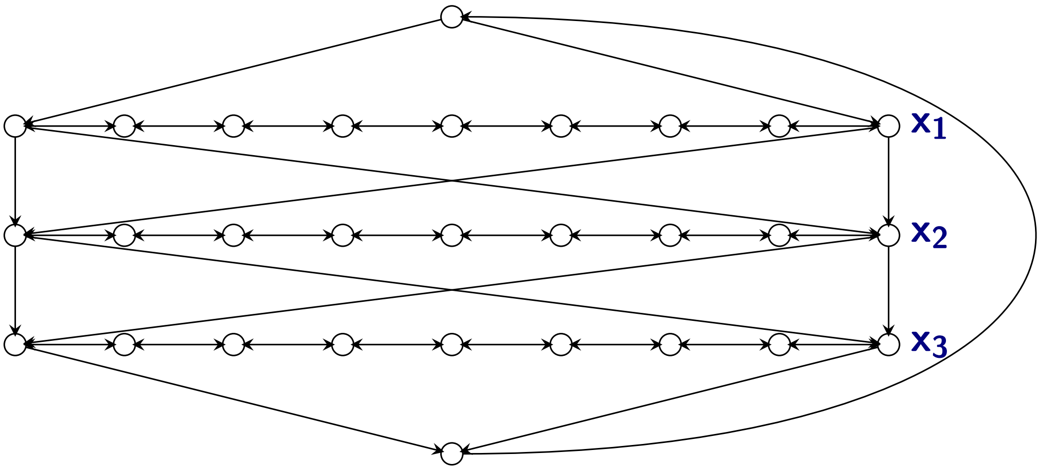
Notation: φ has n variables $x_1, x_2, \ldots, x_n$ and m clauses $C_1, C_2, \ldots, C_m$.

Reduction: First Ideas

- Viewing SAT: Assign values to n variables, and each clauses has 3 ways in which it can be satisfied
- Construct graph with 2^n Hamiltonian cycles, where each cycle corresponds to some boolean assignment
- Then add more graph structure to encode constraints on assignments imposed by the clauses

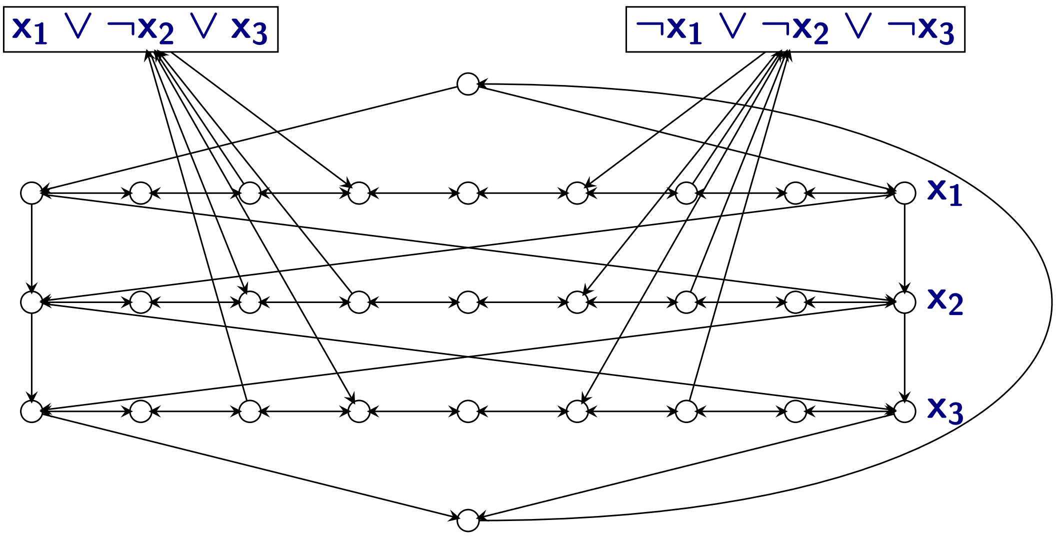
The Reduction: Phase I

- Traverse path i from left to right iff x_i is set to true
- Each path has $3(m+1)$ nodes where m is number of clauses in φ; nodes numbered from left to right (1 to $3m+3$)



The Reduction: Phase II

- Add vertex c_j for clause C_j. c_j has edge *from* vertex $3j$ and *to* vertex $3j+1$ on path i if x_i appears in clause C_j, and has edge *from* vertex $3j+1$ and *to* vertex $3j$ if $\neg x_i$ appears in C_j.



Correctness Proof

Proposition

φ has a satisfying assignment iff $\mathbf{G}_\varphi$ has a Hamiltonian cycle

Proof.

$\Rightarrow$ Let $\mathbf{a}$ be the satisfying assignment for φ. Define Hamiltonian cycle as follows

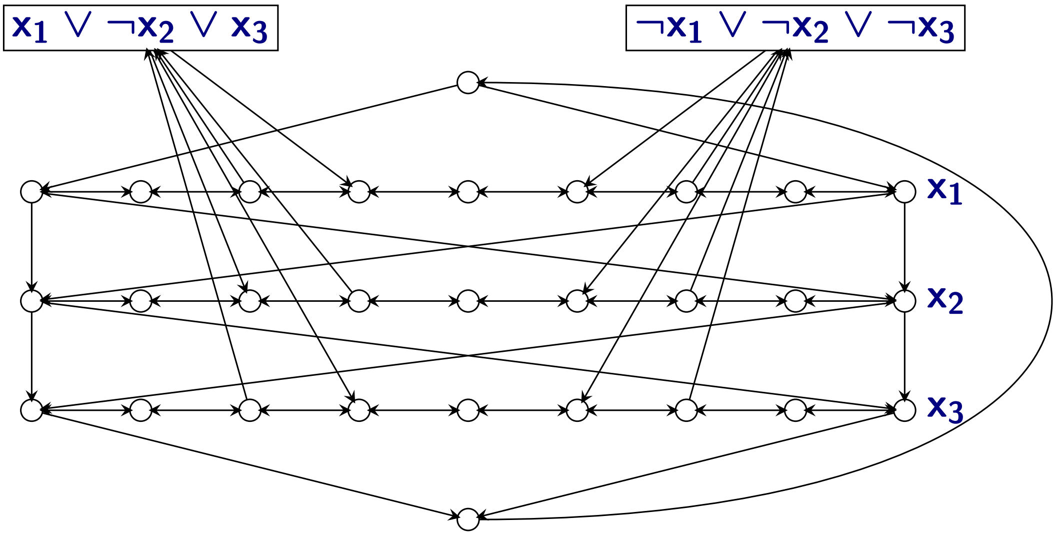
- If $\mathbf{a}(\mathbf{x}_i) = 1$ then traverse path $\mathbf{i}$ from left to right
- If $\mathbf{a}(\mathbf{x}_i) = 0$ then traverse path $\mathbf{i}$ from right to left
- For each clause, path of at least one variable is in the "right" direction to splice in the node corresponding to clause □

Hamiltonian Cycle ⇒ Satisfying assignment

Suppose $\mathbf{\Pi}$ is a Hamiltonian cycle in $\mathbf{G}_\varphi$

- If $\mathbf{\Pi}$ enters $\mathbf{c}_j$ (vertex for clause $\mathbf{C}_j$) from vertex $\mathbf{3j}$ on path $\mathbf{i}$ then it must leave the clause vertex on edge to $\mathbf{3j+1}$ on the *same path* $\mathbf{i}$
  - If not, then only unvisited neighbor of $\mathbf{3j+1}$ on path $\mathbf{i}$ is $\mathbf{3j+2}$
  - Thus, we don't have two unvisited neighbors (one to enter from, and the other to leave) to have a Hamiltonian Cycle
- Similarly, if $\mathbf{\Pi}$ enters $\mathbf{c}_j$ from vertex $\mathbf{3j+1}$ on path $\mathbf{i}$ then it must leave the clause vertex $\mathbf{c}_j$ on edge to $\mathbf{3j}$ on path $\mathbf{i}$

Example



Hamiltonian Cycle $\implies$ Satisfying assignment (contd)

- Thus, vertices visited immediately before and after C_i are connected by an edge
- We can remove c_j from cycle, and get Hamiltonian cycle in $G - c_j$
- Consider Hamiltonian cycle in $G - \{c_1, \ldots c_m\}$; it traverses each path in only one direction, which determines the truth assignment

Hamiltonian Cycle

Problem

Input *Given undirected graph* $\mathbf{G = (V, E)}$

Goal *Does* $\mathbf{G}$ *have a Hamiltonian cycle? That is, is there a cycle that visits every vertex exactly one (except start and end vertex)?*

NP-Completeness

Theorem

Hamiltonian cycle *problem for undirected graphs is* **NP-Complete**.

Proof.

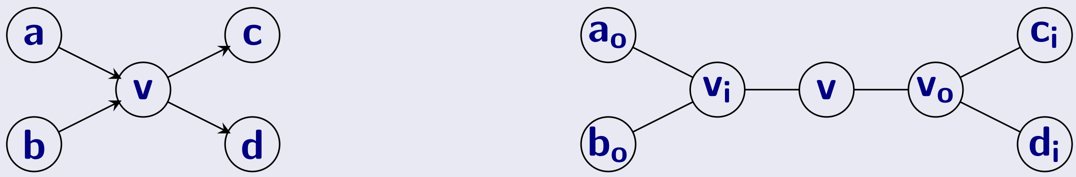
- The problem is in **NP**; proof left as exercise
- Hardness proved by reducing Directed Hamiltonian Cycle to this problem □

Reduction Sketch

Goal: Given directed graph $\mathbf{G}$, need to construct undirected graph $\mathbf{G'}$ such that $\mathbf{G}$ has Hamiltonian Path iff $\mathbf{G'}$ has Hamiltonian path

Reduction

- Replace each vertex $\mathbf{v}$ by 3 vertices: $\mathbf{v_{in}}$, $\mathbf{v}$, and $\mathbf{v_{out}}$
- A directed edge $(\mathbf{a}, \mathbf{b})$ is replaced by edge $(\mathbf{a_{out}}, \mathbf{b_{in}})$



Reduction: Wrapup

- The reduction is polynomial time (exercise)
- The reduction is correct (exercise)

Graph Coloring

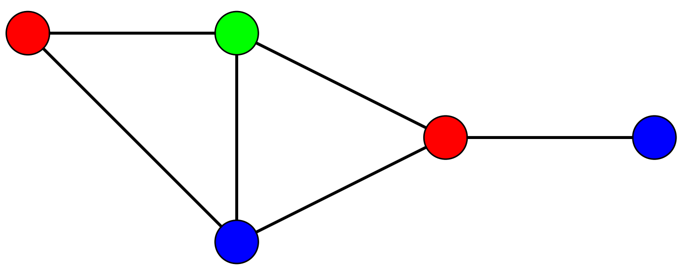
Input Given an undirected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ and integer $\mathbf{k}$

Goal Can the vertices of the graph be colored using $\mathbf{k}$ colors so that vertices connected by an edge do not get the same color?

Graph 3-Coloring

Input Given an undirected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$

Goal Can the vertices of the graph be colored using $\mathbf{3}$ colors so that vertices connected by an edge do not get the same color?



Graph Coloring

Observation: If G is colored with k colors then each color class (nodes of same color) form an independent set in G. Thus, G can be partitioned into k independent sets iff G is k-colorable.

Graph 2-Coloring can be decided in polynomial time.

G is 2-colorable iff G is bipartite! There is a linear time algorithm to check if G is bipartite using **BFS** (see book).

Graph Coloring and Register Allocation

Register Allocation

Assign variables to (at most) k registers such that variables needed at the same time are not assigned to the same register

Interference Graph

Vertices are variables, and there is an edge between two vertices, if the two variables are "live" at the same time.

Observations

- [Chaitin] Register allocation problem is equivalent to coloring the interference graph with k colors
- Moreover, **3-COLOR** $\leq_P$ **k-Register Allocation**, for any $k \geq 3$

Class Room Scheduling

Given $\mathbf{n}$ classes and their meeting times, are $\mathbf{k}$ rooms sufficient?

Reduce to Graph $\mathbf{k}$-Coloring problem

Create graph $\mathbf{G}$

- a node $\mathbf{v_i}$ for each class $\mathbf{i}$
- an edge between $\mathbf{v_i}$ and $\mathbf{v_j}$ if classes $\mathbf{i}$ and $\mathbf{j}$ *conflict*

Exercise: $\mathbf{G}$ is $\mathbf{k}$-colorable iff $\mathbf{k}$ rooms are sufficient

Frequency Assignments in Cellular Networks

Cellular telephone systems that use Frequency Division Multiple Access (FDMA) (example: GSM in Europe and Asia and AT&T in USA)

- Breakup a frequency range $[a, b]$ into disjoint *bands* of frequencies $[a_0, b_0], [a_1, b_1], \ldots, [a_k, b_k]$
- Each cell phone tower (simplifying) gets one band
- Constraint: nearby towers cannot be assigned same band, otherwise signals will interference

Problem: given $\mathbf{k}$ bands and some region with $\mathbf{n}$ towers, is there a way to assign the bands to avoid interference?

Can reduce to $\mathbf{k}$-coloring by creating intereference/conflict graph on towers

3-Coloring is **NP**-Complete

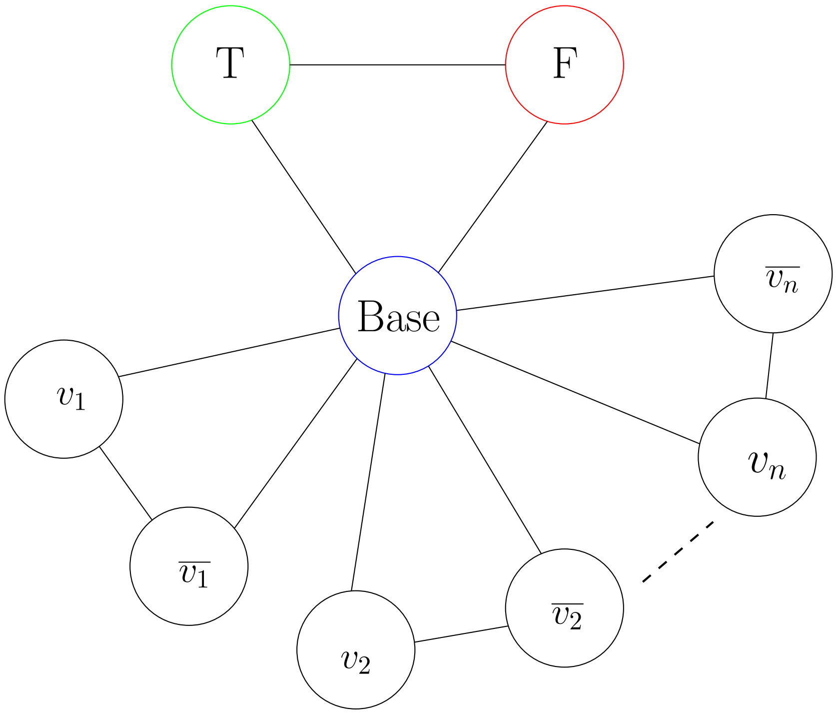
- 3-Coloring is in **NP**
  - Certificate: for each node a color from $\{1, 2, 3\}$
  - Certifier: Check if for each edge (u, v), the color of u is different from that of v
- Hardness: We will show **3-SAT** $\leq_P$ **3-Coloring**

Reduction Idea

Start with **3SAT** formula (i.e., $3\mathrm{CNF}$ formula) φ with n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$. Create graph G_φ such that G_φ is 3-colorable iff φ is satisfiable

- need to establish truth assignment for $x_1, \ldots, x_n$ via colors for some nodes in G_φ.
- create triangle with node True, False, Base
- for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
- If graph is 3-colored, either v_i or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to v_i
- Need to add constraints to ensure clauses are satisfied (next phase)

Figure

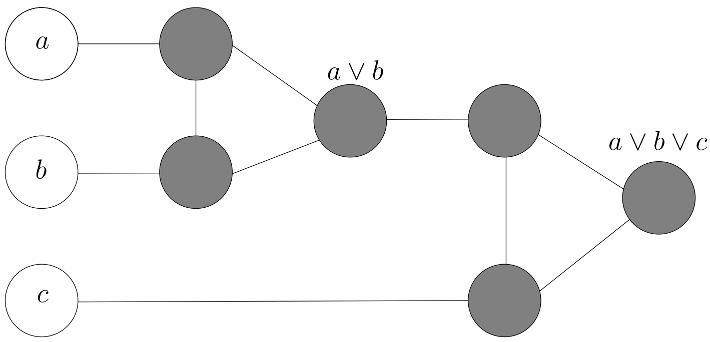


Clause Satisfiability Gadget

For each clause $\mathbf{C_j = (a \vee b \vee c)}$, create a small gadget graph

- gadget graph connects to nodes corresponding to **a, b, c**
- needs to implement OR

OR-gadget-graph:



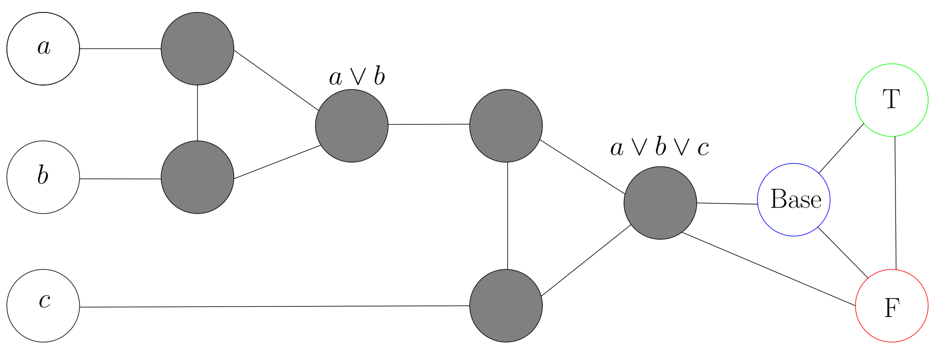
OR-Gadget Graph

Property: if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are colored False in a 3-coloring then output node of OR-gadget has to be colored False.

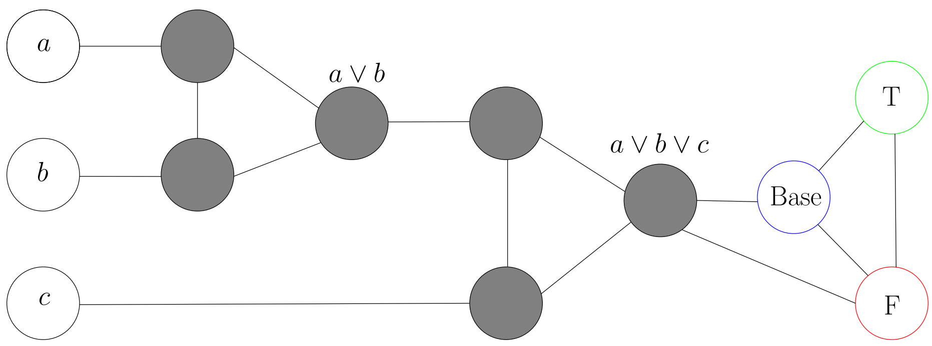
Property: if one of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is colored True then OR-gadget can be 3-colored such that output node of OR-gadget is colored True.

Reduction

- create triangle with nodes True, False, Base
- for each variable $\mathbf{x_i}$ two nodes $\mathbf{v_i}$ and $\bar{\mathbf{v}}_\mathbf{i}$ connected in a triangle with common Base
- for each clause $\mathbf{C_j} = (\mathbf{a} \vee \mathbf{b} \vee \mathbf{c})$, add OR-gadget graph with input nodes $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and connect output node of gadget to both False and Base



Reduction



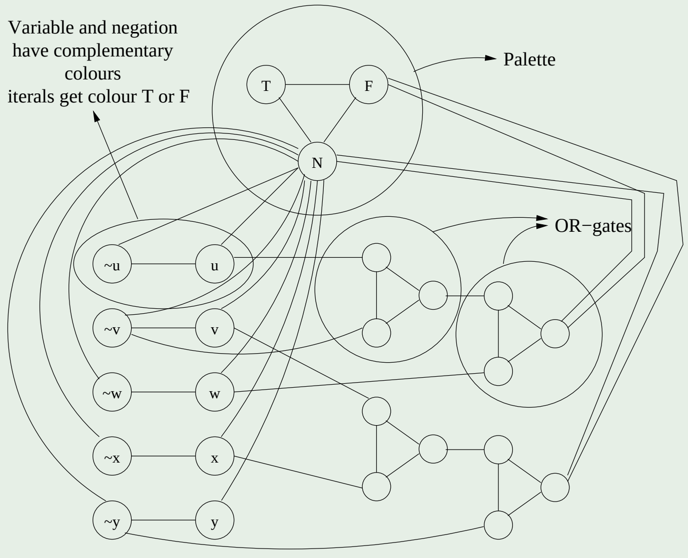
Claim

No legal **3***-coloring of above graph (with coloring of nodes* **T, F, B** *fixed) in which* **a, b, c** *are colored False. If any of* **a, b, c** *are colored True then there is a legal* **3***-coloring of above graph.*

Reduction Outline

Example

$\varphi = (\mathbf{u} \lor \neg \mathbf{v} \lor \mathbf{w}) \land (\mathbf{v} \lor \mathbf{x} \lor \neg \mathbf{y})$



Correctness of Reduction

φ is satisfiable implies $\mathbf{G}_\varphi$ is 3-colorable

- if $\mathbf{x_i}$ is assigned True, color $\mathbf{v_i}$ True and $\bar{\mathbf{v}}_\mathbf{i}$ False
- for each clause $\mathbf{C_j = (a \lor b \lor c)}$ at least one of $\mathbf{a, b, c}$ is colored True. OR-gadget for $\mathbf{C_j}$ can be 3-colored such that output is True.

$\mathbf{G}_\varphi$ is 3-colorable implies φ is satisfiable

- if $\mathbf{v_i}$ is colored True then set $\mathbf{x_i}$ to be True, this is a legal truth assignment
- consider any clause $\mathbf{C_j = (a \lor b \lor c)}$. it cannot be that all $\mathbf{a, b, c}$ are False. If so, output of OR-gadget for $\mathbf{C_j}$ has to be colored False but output is connected to Base and False!

Other NP-Complete Problems

- 3-Dimensional Matching
- Subset Sum

Read book.

Need to Know NP-Complete Problems

- 3-SAT
- Circuit-SAT
- Independent Set
- Vertex Cover
- Clique
- Set Cover
- Hamiltonian Cycle in Directed/Undirected Graphs
- 3-Coloring
- 3-D Matching
- Subset Sum

Subset Sum and Knapsack

Subset Sum Problem: Given n integers $a_1, a_2, \ldots, a_n$ and a target B, is there a subset of S of $\{a_1, \ldots, a_n\}$ such that the numbers in S add up *precisely* to B?

Subset Sum is **NP-Complete**— see book.

Knapsack: Given n items with item i having size s_i and profit p_i, a knapsack of capacity B, and a target profit P, is there a subset S of items that can be packed in the knapsack and the profit of S is at least P?

Show Knapsack problem is **NP-Complete** via reduction from Subset Sum (exercise).

Subset Sum and Knapsack

Subset Sum can be solved in $O(nB)$ time using dynamic programming (exercise).

Implies that problem is hard only when numbers $a_1, a_2, \ldots, a_n$ are exponentially large compared to n. That is, each a_i requires polynomial in n bits.

Number problems of the above type are said to be **weakly NP-Complete**.