NP Completeness and Cook-Levin Theorem

Lecture 22

April 19, 2011

P and NP and Turing Machines

- **P**: set of decision problems that have polynomial time algorithms
- **NP**: set of decision problems that have polynomial time non-deterministic algorithms

Question: What is an algorithm? Depends on the model of computation!

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Formally speaking our model of computation is Turing Machines.

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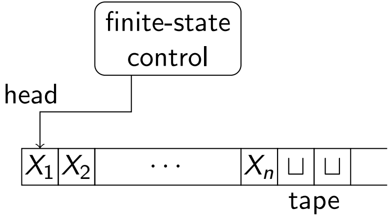
Turing Machines: Recap



- Infinite tape
- Finite state control
- Input at beginning of tape
- Special tape letter "blank" $\sqcup$
- Head can move only one cell to left or right

Turing Machines: Formally

A TM $\mathbf{M} = (\mathbf{Q}, \mathbf{\Sigma}, \mathbf{\Gamma}, \delta, \mathbf{q_0}, \mathbf{q_{accept}}, \mathbf{q_{reject}})$:

- $\mathbf{Q}$ is set of states in finite control
- $\mathbf{q_0}$ start state, $\mathbf{q_{accept}}$ is accept state, $\mathbf{q_{reject}}$ is reject state
- $\mathbf{\Sigma}$ is input alphabet, $\mathbf{\Gamma}$ is tape alphabet (includes $\sqcup$)
- $\delta : \mathbf{Q} \times \mathbf{\Gamma} \rightarrow \{\mathbf{L}, \mathbf{R}\} \times \mathbf{\Gamma} \times \mathbf{Q}$ is transition function
  - $\delta(\mathbf{q}, \mathbf{a}) = (\mathbf{q'}, \mathbf{b}, \mathbf{L})$ means that $\mathbf{M}$ in state $\mathbf{q}$ and head seeing $\mathbf{a}$ on tape will move to state $\mathbf{q'}$ while replacing $\mathbf{a}$ on tape with $\mathbf{b}$ and head moves left.

$\mathbf{L(M)}$: language accepted by $\mathbf{M}$ is set of all input strings $\mathbf{s}$ on which $\mathbf{M}$ accepts; that is:

- TM is started in state $\mathbf{q_0}$.
- Initially, the tape head is located at the first cell.
- The tape contain $\mathbf{s}$ on the tape followed by blanks.
- The TM halts in the state $\mathbf{q_{accept}}$.

P via TMs

Definition

$\mathbf{M}$ is a polynomial time TM if there is some polynomial $\mathbf{p}(\cdot)$ such that on all inputs $\mathbf{w}$, $\mathbf{M}$ halts in $\mathbf{p}(|\mathbf{w}|)$ steps.

Definition

$\mathbf{L}$ is a language in $\mathbf{P}$ iff there is a polynomial time TM $\mathbf{M}$ such that $\mathbf{L} = \mathbf{L}(\mathbf{M})$.

NP via TMs

Definition

$\mathbf{L}$ is an **NP** language iff there is a *non-deterministic* polynomial time TM $\mathbf{M}$ such that $\mathbf{L} = \mathbf{L}(\mathbf{M})$.

Non-deterministic TM: each step has a choice of moves

- $\delta : \mathbf{Q} \times \Gamma \rightarrow \mathcal{P}(\mathbf{Q} \times \Gamma \times \{\mathbf{L}, \mathbf{R}\})$.
    - Example: $\delta(\mathbf{q}, \mathbf{a}) = \{(\mathbf{q}_1, \mathbf{b}, \mathbf{L}), (\mathbf{q}_2, \mathbf{c}, \mathbf{R}), (\mathbf{q}_3, \mathbf{a}, \mathbf{R})\}$ means that $\mathbf{M}$ can non-deterministically choose one of the three possible moves from $(\mathbf{q}, \mathbf{a})$.
- $\mathbf{L}(\mathbf{M})$: set of all strings $\mathbf{s}$ on which there *exists* some sequence of valid choices at each step that lead from $\mathbf{q}_0$ to $\mathbf{q}_{\mathrm{accept}}$

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Non-deterministic TM: each step has a choice of moves

- $\delta : Q \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$.
    - Example: $\delta(q, a) = \{(q_1, b, L), (q_2, c, R), (q_3, a, R)\}$ means that $\mathbf{M}$ can non-deterministically choose one of the three possible moves from (q, a).
- $\mathbf{L}(\mathbf{M})$: set of all strings s on which there *exists* some sequence of valid choices at each step that lead from q_0 to q_{accept}

Non-deterministic TMs vs certifiers

Two definition of **NP**:

- **L** is in **NP** iff **L** has a polynomial time certifier $\mathbf{C}(\cdot, \cdot)$.
- **L** is in **NP** iff **L** is decided by a non-deterministic polynomial time TM **M**.

Claim: Two definitions are equivalent. Why?

Informal proof idea: the certificate **t** for **C** corresponds to non-deterministic choices of **M** and vice-versa.
In other words **L** is in **NP** iff **L** is accepted by a NTM which first guesses a proof **t** of length poly in input $|\mathbf{s}|$ and then acts as a *deterministic* TM.

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Non-determinism, guessing and verification

- A non-deterministic machine has choices at each step and accepts a string if there *exists* a set of choices which lead to a final state.
- Equivalently the choices can be thought of as *guessing* a solution and then *verifying* that solution. In this view all the choices are made a priori and hence the verification can be deterministic. The "guess" is the "proof" and the "verifier" is the "certifier".
- We reemphasize the asymmetry inherent in the definition of non-determinism. Strings in the language can be easily verified. No easy way to verify that a string is not in the language.

Algorithms: TMs vs RAM Model

Why do we use TMs some times and RAM Model other times?

- TMs are very simple: no complicated instruction set, no jumps/pointers, no explicit loops etc.
  - Simplicity is useful in proofs.
  - The “right” formal bare-bones model when dealing with subtleties.
- RAM model is a closer approximation to the running time/space usage of realistic computers for reasonable problem sizes
  - Not appropriate for certain kinds of formal proofs when algorithms can take super-polynomial time and space

"Hardest" Problems

Question

What is the hardest problem in **NP**? How do we define it?

Towards a definition

- Hardest problem must be in **NP**
- Hardest problem must be at least as "difficult" as every other problem in **NP**

NP-Complete Problems

Definition

A problem $\mathbf{X}$ is said to be **NP-Complete** if

- $\mathbf{X} \in \mathbf{NP}$
- (Hardness) For any $\mathbf{Y} \in \mathbf{NP}$, $\mathbf{Y} \leq_{\mathbf{P}} \mathbf{X}$

Solving **NP-Complete** Problems

Proposition

Suppose **X** *is* **NP-Complete**. *Then* **X** *can be solved in polynomial time iff* $\mathbf{P} = \mathbf{NP}$

Proof.

$\Rightarrow$ Suppose **X** can be solved in polynomial time

- Let $\mathbf{Y} \in \mathbf{NP}$. We know $\mathbf{Y} \leq_{\mathbf{P}} \mathbf{X}$
- We showed that if $\mathbf{Y} \leq_{\mathbf{P}} \mathbf{X}$ and **X** can be solved in polynomial time, then **Y** can be solved in polynomial time
- Thus, every problem $\mathbf{Y} \in \mathbf{NP}$ is such that $\mathbf{Y} \in \mathbf{P}$; $\mathbf{NP} \subseteq \mathbf{P}$
- Since $\mathbf{P} \subseteq \mathbf{NP}$, we have $\mathbf{P} = \mathbf{NP}$

$\Leftarrow$ Since $\mathbf{P} = \mathbf{NP}$, and $\mathbf{X} \in \mathbf{NP}$, we have a polynomial time algorithm for **X** $\square$

NP-Hard Problems

Definition

A problem **X** is said to be **NP-Hard** if

- (Hardness) For any $\mathbf{Y} \in \mathbf{NP}$, $\mathbf{Y} \leq_{\mathbf{P}} \mathbf{X}$

An **NP-Hard** problem need not be in **NP**!

Example: Halting problem is **NP-Hard** (why?) but not **NP-Complete**.

Consequences of proving NP-Completeness

If **X** is **NP-Complete**

- Since we believe **P** $\neq$ **NP**,
- and solving **X** implies **P** $=$ **NP**.

X is unlikely to be efficiently solvable.

At the very least, many smart people before you have failed to find an efficient algorithm for **X**.

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NP-Complete Problems

Question

Are there any problems that are **NP-Complete**?

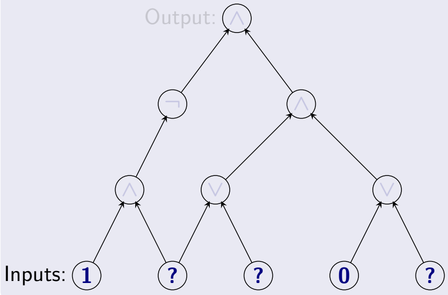
Answer

Yes! Many, many problems are **NP-Complete**.

Circuits

Definition

A circuit is a directed *acyclic* graph with



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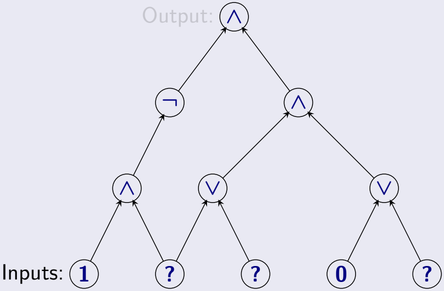

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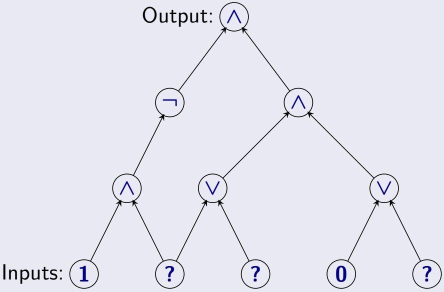

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Cook-Levin Theorem

Definition (Circuit Satisfaction (**CSAT**).)

Given a circuit as input, is there an assignment to the input variables that causes the output to get value **1**?

Theorem (Cook-Levin)

CSAT *is* **NP-Complete**.

Need to show

- **CSAT** is in **NP**
- *every* **NP** problem **X** reduces to **CSAT**.

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Claim

CSAT is in NP.

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CSAT is NP-hard: Idea

Need to show that *every* NP problem X reduces to CSAT.

What does it mean that $X \in NP$?

$X \in NP$ implies that there are polynomials $p()$ and $q()$ and certifier/verifier program C such that for every string s the following is true:

- If s is a YES instance ($s \in X$) then there is a *proof* t of length $p(|s|)$ such that $C(s, t)$ says YES.
- If s is a NO instance ($s \not\in X$) then for every string t of length at $p(|s|)$, $C(s, t)$ says NO.
- $C(s, t)$ runs in time $q(|s| + |t|)$ time (hence polynomial time)

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Reducing X to CSAT

X is in **NP** means we have access to $\mathbf{p}(), \mathbf{q}(), \mathbf{C}(\cdot, \cdot)$.

What is $\mathbf{C}(\cdot, \cdot)$? It is a program or equivalently a Turing Machine!

How are $\mathbf{p}()$ and $\mathbf{q}()$ given? As numbers.

Example: if **3** is given then $\mathbf{p}(\mathbf{n}) = \mathbf{n}^3$.

Thus an **NP** problem is essentially a three tuple $< \mathbf{p}, \mathbf{q}, \mathbf{C} >$ where **C** is either a program or a TM.

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Problem X: Given string $\mathbf{s}$, is $\mathbf{s} \in \mathbf{X}$?

Same as the following: is there a proof $\mathbf{t}$ of length $\mathbf{p}(|\mathbf{s}|)$ such that $\mathbf{C}(\mathbf{s}, \mathbf{t})$ says YES.

How do we reduce **X** to **CSAT**? Need an algorithm $\mathcal{A}$ that

- takes $\mathbf{s}$ (and $< \mathbf{p}, \mathbf{q}, \mathbf{C} >$) and creates a circuit $\mathbf{G}$ in polynomial time in $|\mathbf{s}|$ (note that $< \mathbf{p}, \mathbf{q}, \mathbf{C} >$ are fixed).
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Simple but Big Idea: Programs are essentially the same as Circuits!

- Convert $C(s, t)$ into a circuit G with t as unknown inputs (rest is known including s)
- We know that $|t| = p(|s|)$ so express boolean string t as $p(|s|)$ variables $t_1, t_2, \ldots, t_k$ where $k = p(|s|)$.
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Reducing $\mathbf{X}$ to CSAT

How do we reduce $\mathbf{X}$ to **CSAT**? Need an algorithm $\mathcal{A}$ that

- takes $\mathbf{s}$ (and $< \mathbf{p}, \mathbf{q}, \mathbf{C} >$) and creates a circuit $\mathbf{G}$ in polynomial time in $|\mathbf{s}|$ (note that $< \mathbf{p}, \mathbf{q}, \mathbf{C} >$ are fixed).
- $\mathbf{G}$ is satisfiable if and only if there is a proof $\mathbf{t}$ such that $\mathbf{C(s, t)}$ says YES

Simple but Big Idea: Programs are essentially the same as Circuits!

- Convert $\mathbf{C(s, t)}$ into a circuit $\mathbf{G}$ with $\mathbf{t}$ as unknown inputs (rest is known including $\mathbf{s}$)
- We know that $|\mathbf{t}| = \mathbf{p}(|\mathbf{s}|)$ so express boolean string $\mathbf{t}$ as $\mathbf{p}(|\mathbf{s}|)$ variables $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_k$ where $\mathbf{k} = \mathbf{p}(|\mathbf{s}|)$.
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Example: Independent Set

- Problem: Does $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ have an **Independent Set** of size $\geq \mathbf{k}$?
  - Certificate: Set $\mathbf{S} \subseteq \mathbf{V}$
  - Certifier: Check $|\mathbf{S}| \geq \mathbf{k}$ and no pair of vertices in $\mathbf{S}$ is connected by an edge

Formally, why is **Independent Set** in **NP**?

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Example: Independent Set

Formally why is **Independent Set** in **NP**?

- Input:
  $< \mathbf{n}, \mathbf{y}_{1,1}, \mathbf{y}_{1,2}, \ldots, \mathbf{y}_{1,n}, \mathbf{y}_{2,1}, \ldots, \mathbf{y}_{2,n}, \ldots, \mathbf{y}_{n,1}, \ldots, \mathbf{y}_{n,n}, \mathbf{k} >$ encodes $< \mathbf{G}, \mathbf{k} >$.
  - $\mathbf{n}$ is number of vertices in $\mathbf{G}$
  - $\mathbf{y}_{i,j}$ is a bit which is $\mathbf{1}$ if edge $\mathbf{(i, j)}$ is in $\mathbf{G}$ and $\mathbf{0}$ otherwise (adjacency matrix representation)
  - $\mathbf{k}$ is size of independent set
- Certificate: $\mathbf{t} = \mathbf{t}_1\mathbf{t}_2 \ldots \mathbf{t}_n$. Interpretation is that $\mathbf{t}_i$ is $\mathbf{1}$ if vertex $\mathbf{i}$ is in the independent set, $\mathbf{0}$ otherwise.

Certifier for Independent Set

Certifier $C(s, t)$ for **Independent Set**:

```
if (t_1 + t_2 + ... + t_n < k) then
    return NO
else
    for each (i, j) do
        if (t_i ∧ t_j ∧ y_{i,j}) then
            return NO

return YES
```

$$\textbf{if } (t_1 + t_2 + \ldots + t_n < k) \textbf{ then}$$

Example: Independent Set

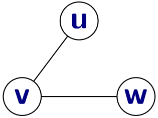
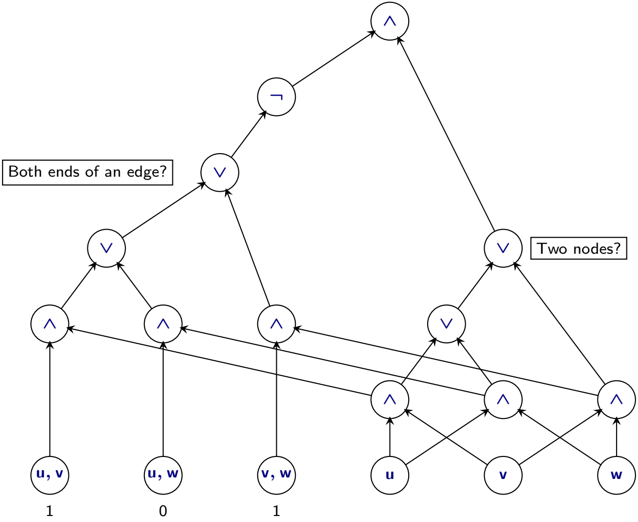


Figure: Graph **G** with **k = 2**



Circuit from Certifier

Programs, Turing Machines and Circuits

Consider "program" $\mathbf{A}$ that takes $\mathbf{f(|s|)}$ steps on input string $\mathbf{s}$.

Question: What computer is the program running on and what does *step* mean?

Real computers difficult to reason with mathematically because

- instruction set is too rich
- pointers and control flow jumps in one step
- assumption that pointer to code fits in one word

Turing Machines

- simpler model of computation to reason with
- can simulate real computers with *polynomial* slow down
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Certifiers that at TMs

Assume $\mathbf{C}(\cdot, \cdot)$ is a (deterministic) Turing Machine $\mathbf{M}$

Problem: Given $\mathbf{M}$, input $\mathbf{s}$, $\mathbf{p}$, $\mathbf{q}$ decide if there is a proof $\mathbf{t}$ of length $\mathbf{p}(|\mathbf{s}|)$ such that $\mathbf{M}$ on $\mathbf{s}, \mathbf{t}$ will halt in $\mathbf{q}(|\mathbf{s}|)$ time and say YES.

There is an algorithm $\mathcal{A}$ that can reduce above problem to **CSAT** mechanically as follows.

- $\mathcal{A}$ first computes $\mathbf{p}(|\mathbf{s}|)$ and $\mathbf{q}(|\mathbf{s}|)$.
- Knows that $\mathbf{M}$ can use at most $\mathbf{q}(|\mathbf{s}|)$ memory/tape cells
- Knows that $\mathbf{M}$ can run for at most $\mathbf{q}(|\mathbf{s}|)$ time
- Simulates the evolution of the state of $\mathbf{M}$ and memory over time using a big circuit.

Simulation of Computation via Circuit

- Think of $\mathbf{M}$'s state at time ℓ as a string $\mathbf{x}^{\ell} = \mathbf{x_1 x_2 \ldots x_k}$ where each $\mathbf{x_i} \in \{\mathbf{0}, \mathbf{1}, \mathbf{B}\} \times \mathbf{Q} \cup \{\mathbf{q_{-1}}\}$.
- At time $\mathbf{0}$ the state of $\mathbf{M}$ consists of input string $\mathbf{s}$ a guess $\mathbf{t}$ (unknown variables) of length $\mathbf{p(|s|)}$ and rest $\mathbf{q(|s|)}$ blank symbols.
- At time $\mathbf{q(|s|)}$ we wish to know if $\mathbf{M}$ stops in $\mathbf{q_{accept}}$ with say all blanks on the tape.
- We write a circuit $\mathbf{C}_{\ell}$ which captures the transition of $\mathbf{M}$ from time ℓ to time $\ell + \mathbf{1}$.
- Composition of the circuits for all times $\mathbf{0}$ to $\mathbf{q(|s|)}$ gives a big (still poly) sized circuit $\mathcal{C}$
- The final output of $\mathcal{C}$ should be true if and only if the entire state of $\mathbf{M}$ at the end leads to an accept state.

NP-Hardness of Circuit Satisfaction

Key Ideas in reduction:

- Use TMs as the code for certifier for simplicity
- Since **p()** and **q()** are known to $\mathcal{A}$, it can set up all required memory and time steps in advance
- Simulate computation of the TM from one time to the next as a circuit that only looks at three adjacent cells at a time

Note: Above reduction can be done to **SAT** as well. Reduction to **SAT** was the original proof of Steve Cook.

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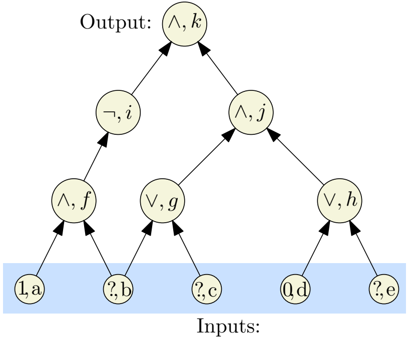
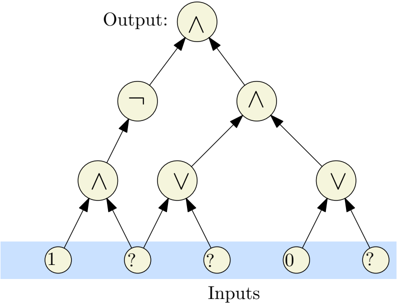
SAT is NP-Complete

- We have seen that **SAT** $\in$ **NP**
- To show **NP-Hardness**, we will reduce Circuit Satisfiability (**CSAT**) to **SAT**
  Instance of **CSAT** (we label each node):

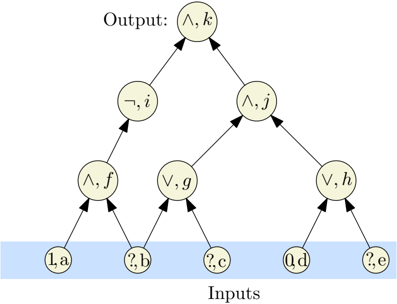


Converting a circuit into a CNF formula

Label the nodes



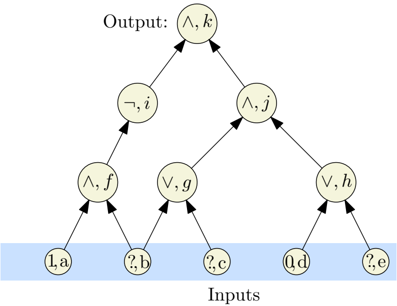
(A) Input circuit



(B) Label the nodes.

Converting a circuit into a CNF formula

Introduce a variable for each node



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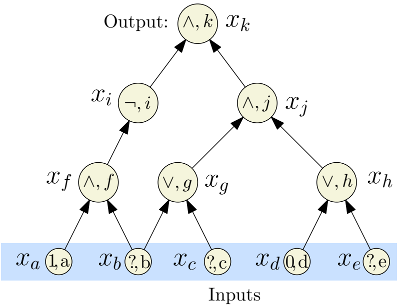
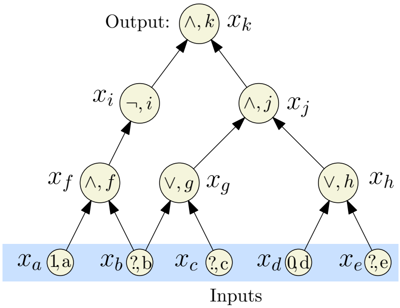


(C) Introduce var for each node.

Converting a circuit into a CNF formula

Write a sub-formula for each variable that is true if the var is computed correctly.



(C) Introduce var for each node.

$\mathbf{x_k}$ (Demand a sat' assignment!)

$\mathbf{x_k = x_i \wedge x_k}$

$\mathbf{x_j = x_g \wedge x_h}$

$\mathbf{x_i = \neg x_f}$

$\mathbf{x_h = x_d \vee x_e}$

$\mathbf{x_g = x_b \vee x_c}$

$\mathbf{x_f = x_a \wedge x_b}$

$\mathbf{x_d = 0}$

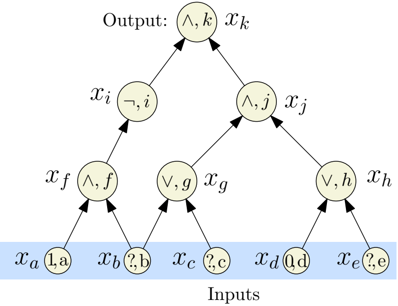
$\mathbf{x_a = 1}$

(D) Write a sub-formula for each variable that is true if the var is computed correctly.

Converting a circuit into a CNF formula

Convert each sub-formula to an equivalent CNF formula

x_k	x_k
$x_k = x_i \wedge x_j$	$(\neg x_k \vee x_i) \wedge (\neg x_k \vee x_j) \wedge (x_k \vee \neg x_i \vee \neg x_j)$
$x_j = x_g \wedge x_h$	$(\neg x_j \vee x_g) \wedge (\neg x_j \vee x_h) \wedge (x_j \vee \neg x_g \vee \neg x_h) \wedge$
$x_i = \neg x_f$	$(x_i \vee x_f) \wedge (\neg x_i \vee x_f) \wedge$
$x_h = x_d \vee x_e$	$(x_h \vee \neg x_d) \wedge (x_h \vee \neg x_e) \wedge (\neg x_h \vee x_d \vee x_e)$
$x_g = x_b \vee x_c$	$(x_g \vee \neg x_b) \wedge (x_g \vee \neg x_c) \wedge (\neg x_g \vee x_b \vee x_c)$
$x_f = x_a \wedge x_b$	$(\neg x_f \vee x_a) \wedge (\neg x_f \vee x_b) \wedge (x_f \vee \neg x_a \vee \neg x_b)$
$x_d = 0$	$\neg x_d$
$x_a = 1$	x_a



$$
\begin{aligned}
& x_k \wedge (\neg x_k \vee x_i) \wedge (\neg x_k \vee x_j) \\
& \wedge (x_k \vee \neg x_i \vee \neg x_j) \wedge (\neg x_j \vee x_g) \\
& \wedge (\neg x_j \vee x_h) \wedge (x_j \vee \neg x_g \vee \neg x_h) \\
& \wedge (x_i \vee x_f) \wedge (\neg x_i \vee x_f) \\
& \wedge (x_h \vee \neg x_d) \wedge (x_h \vee \neg x_e) \\
& \wedge (\neg x_h \vee x_d \vee x_e) \wedge (x_g \vee \neg x_b) \\
& \wedge (x_g \vee \neg x_c) \wedge (\neg x_g \vee x_b \vee x_c) \\
& \wedge (\neg x_f \vee x_a) \wedge (\neg x_f \vee x_b) \\
& \wedge (x_f \vee \neg x_a \vee \neg x_b) \wedge (\neg x_d) \wedge x_a
\end{aligned}
$$

We got a CNF formula that is satisfiable if and only if the original circuit is satisfiable.

Reduction: CSAT $\leq_P$ SAT

- For each gate (vertex) $\mathbf{v}$ in the circuit, create a variable $\mathbf{x_v}$
- Case $\neg$: $\mathbf{v}$ is labeled $\neg$ and has one incoming edge from $\mathbf{u}$ (so $\mathbf{x_v = \neg x_u}$). In **SAT** formula generate, add clauses $(\mathbf{x_u} \vee \mathbf{x_v})$, $(\neg \mathbf{x_u} \vee \neg \mathbf{x_v})$. Observe that

$$\mathbf{x_v} = \neg \mathbf{x_u} \text{ is true} \quad \Longleftrightarrow \quad \begin{array}{l} (\mathbf{x_u} \vee \mathbf{x_v}) \\ (\neg \mathbf{x_u} \vee \neg \mathbf{x_v}) \end{array} \text{ both true.}$$

Reduction: CSAT $\leq_P$ SAT

Continued...

- Case $\vee$: So $x_v = x_u \vee x_w$. In SAT formula generated, add clauses $(x_v \vee \neg x_u)$, $(x_v \vee \neg x_w)$, and $(\neg x_v \vee x_u \vee x_w)$. Again, observe that

$$x_v = x_u \vee x_w \text{ is true} \quad \iff \quad \begin{array}{l} (x_v \vee \neg x_u), \\ (x_v \vee \neg x_w), \\ (\neg x_v \vee x_u \vee x_w) \end{array} \quad \text{all true.}$$

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$$x_v = x_u \wedge x_w \text{ is true} \iff \begin{matrix} (\neg x_v \vee x_u), \\ (\neg x_v \vee x_w), \\ (x_v \vee \neg x_u \vee \neg x_w) \end{matrix} \quad \text{all true.}$$

Reduction: CSAT $\leq_P$ SAT

Continued...

- If v is an input gate with a fixed value then we do the following. If $x_v = 1$ add clause x_v. If $x_v = 0$ add clause $\neg x_v$
- Add the clause x_v where v is the variable for the output gate

Correctness of Reduction

Need to show circuit $\mathbf{C}$ is satisfiable iff $\varphi_{\mathbf{C}}$ is satisfiable

$\Rightarrow$ Consider a satisfying assignment $\mathbf{a}$ for $\mathbf{C}$

- Find values of all gates in $\mathbf{C}$ under $\mathbf{a}$
- Give value of gate $\mathbf{v}$ to variable $\mathbf{x_v}$; call this assignment $\mathbf{a'}$
- $\mathbf{a'}$ satisfies $\varphi_{\mathbf{C}}$ (exercise)

$\Leftarrow$ Consider a satisfying assignment $\mathbf{a}$ for $\varphi_{\mathbf{C}}$

- Let $\mathbf{a'}$ be the restriction of $\mathbf{a}$ to only the input variables
- Value of gate $\mathbf{v}$ under $\mathbf{a'}$ is the same as value of $\mathbf{x_v}$ in $\mathbf{a}$
- Thus, $\mathbf{a'}$ satisfies $\mathbf{C}$

Theorem

SAT *is* **NP-Complete**.

Proving that a problem X is NP-Complete

To prove X is **NP-Complete**, show

- Show X is in **NP**.
  - certificate/proof of polynomial size in input
  - polynomial time certifier $C(s, t)$
- Reduction from a known **NP-Complete** problem such as **CSAT** or **SAT** to X

SAT $\leq_P$ X implies that every **NP** problem $Y \leq_P X$. Why?
Transitivity of reductions:

$Y \leq_P$ SAT and SAT $\leq_P X$ and hence $Y \leq_P X$.

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NP-Completeness via Reductions

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- **CSAT** $\leq_P$ **SAT** and **SAT** is in **NP** and hence SAT is **NP-Complete**
- **SAT** $\leq_P$ **3-SAT** and hence 3-SAT is **NP-Complete**
- **3-SAT** $\leq_P$ Independent Set (which is in **NP**) and hence Independent Set is **NP-Complete**
- Vertex Cover is **NP-Complete**
- Clique is **NP-Complete**

Hundreds and thousands of different problems from many areas of science and engineering have been shown to be **NP-Complete**.

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