Chapter 21

Reductions and NP

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21.1 Reductions Continued

21.1.0.1 Polynomial Time Reduction

A polynomial time reduction from a decision problem $X$ to a decision problem $Y$ is an algorithm $A$ that has the following properties:

- given an instance $I_X$ of $X$, $A$ produces an instance $I_Y$ of $Y$
- $A$ runs in time polynomial in $|I_X|$. This implies that $|I_Y|$ (size of $I_Y$) is polynomial in $|I_X|$
- Answer to $I_X$ YES iff answer to $I_Y$ is YES.

Notation: $X \leq_P Y$ if $X$ reduces to $Y$

Proposition 21.1.1 If $X \leq_P Y$ then a polynomial time algorithm for $Y$ implies a polynomial time algorithm for $X$.

Such a reduction is called a Karp reduction. Most reductions we will need are Karp reductions

21.1.0.2 A More General Reduction

Turing Reduction (the one given in the book)

Problem $X$ polynomial time reduces to $Y$ if there is an algorithm $A$ for $X$ that has the following properties:
• on any given instance $I_X$ of $X$, $A$ uses polynomial in $|I_X|$ “steps”
• a step is either a standard computation step or
• a sub-routine call to an algorithm that solves $Y$

Note: In making sub-routine call to algorithm to solve $Y$, $A$ can only ask questions of size polynomial in $|I_X|$. Why?
Above reduction is called a Turing reduction.

21.1.0.3 Example of Turing Reduction

Input: Collection of arcs on a circle.
Goal: Compute the maximum number of non-overlapping arcs.

Reduced to the following problem:

Input: Collection of intervals on the line.
Goal: Compute the maximum number of non-overlapping intervals.

How? Used algorithm for interval problem multiple times.

21.1.0.4 Turing vs Karp Reductions

• Turing reductions more general than Karp reductions
• Turing reduction useful in obtaining algorithms via reductions
• Karp reduction is simpler and easier to use to prove hardness of problems
• Perhaps surprisingly, Karp reductions, although limited, suffice for most known NP-Completeness proofs

21.1.1 The Satisfiability Problem (SAT)

21.1.1.1 Propositional Formulas

Definition 21.1.2 Consider a set of boolean variables $x_1, x_2, \ldots x_n$

• A literal is either a boolean variable $x_i$ or its negation $\neg x_i$
• A clause is a disjunction of literals. For example, $x_1 \lor x_2 \lor \neg x_4$ is a clause
• A formula in conjunctive normal form (CNF) is propositional formula which is a conjunction of clauses
  
  \[ (x_1 \lor x_2 \lor \neg x_4) \land (x_2 \lor \neg x_3) \land x_5 \] is a formula in CNF
A formula $\varphi$ is in 3CNF if it is a CNF formula such that every clause has exactly 3 literals

$$(x_1 \lor x_2 \lor \neg x_4) \land (x_2 \lor \neg x_3 \lor x_1)$$

is a 3CNF formula, but

$$(x_1 \lor x_2 \lor \neg x_4) \land (x_2 \lor \neg x_3) \land x_5$$

is not.

### 21.1.1.2 Satisfiability

**SAT**

Given a CNF formula $\varphi$, is there a truth assignment to variables such that $\varphi$ evaluates to true?

**Example 21.1.3**

$$(x_1 \lor x_2 \lor \neg x_4) \land (x_2 \lor \neg x_3) \land x_5 \text{ is satisfiable; take } x_1, x_2, \ldots, x_5 \text{ to be all true}$$

$$(x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2) \land (x_1 \lor x_2) \text{ is not satisfiable}$$

**3-SAT**

Given a 3CNF formula $\varphi$, is there a truth assignment to variables such that $\varphi$ evaluates to true?

### 21.1.1.3 Importance of SAT and 3-SAT

- SAT and 3-SAT are basic constraint satisfaction problems
- Many different problems can be reduced to them because of the simple yet powerful expressivity of logical constraints
- Arise naturally in many applications involving hardware and software verification and correctness
- As we will see, it is a fundamental problem in the theory of NP-Completeness

### 21.1.2 Sat and 3-SAT

#### 21.1.2.1 SAT $\leq P$ 3-SAT

Easy to see that 3-SAT $\leq P$ SAT. A 3-SAT instance is also an instance of SAT.

We can show that SAT $\leq P$ 3-SAT.

Given $\varphi$ a SAT formula we create a 3-SAT formula $\varphi'$ such that

- $\varphi$ is satisfiable iff $\varphi'$ is satisfiable
- $\varphi'$ can be constructed from $\varphi$ in time polynomial in $|\varphi|$.

**Idea:** if a clause of $\varphi$ is not of length 3, replace it with several clauses of length exactly 3
21.1.2.2 SAT \leq_p 3-SAT

Reduction Ideas

Challenge: Some of the clauses in \( \phi \) may have less or more than 3 literals. For each clause with < 3 or > 3 literals, we will construct a set of logically equivalent clauses.

- Case clause with 1 literal: Let \( c = \ell \). Let \( u, v \) be new variables. Consider

  \[
  c' = (\ell \lor u \lor v) \land (\ell \lor u \lor \neg v) \land (\ell \lor \neg u \lor v) \land (\ell \lor \neg u \lor \neg v)
  \]

  Observe that \( c' \) is satisfiable iff \( c \) is satisfiable

21.1.2.3 SAT \leq_p 3-SAT (contd)

Reduction Ideas: 2 and more literals

- Case clause with 2 literals: Let \( c = \ell_1 \lor \ell_2 \). Let \( u \) be a new variable. Consider

  \[
  c' = (\ell_1 \lor \ell_2 \lor u) \land (\ell_1 \lor \ell_2 \lor \neg u)
  \]

  Again \( c \) is satisfiable iff \( c' \) is satisfiable

21.1.2.4 SAT \leq_p 3-SAT (contd)

Reduction Ideas: 2 and more literals

- Case clause with > 3 literals: Let \( c = \ell_1 \lor \cdots \lor \ell_k \). Let \( u_1, \ldots, u_{k-3} \) be new variables. Consider

  \[
  c' = (\ell_1 \lor \ell_2 \lor u_1) \land (\ell_3 \lor \neg u_1 \lor u_2) \land (\ell_4 \lor \neg u_2 \lor u_3) \land \\
  \cdots \land (\ell_{k-2} \lor \neg u_{k-4} \lor u_{k-3}) \land (\ell_{k-1} \lor \ell_k \lor \neg u_{k-3})
  \]

  \( c \) is satisfiable iff \( c' \) is satisfiable

  Another way to see it — reduce size of clause by one:

  \[
  c' = (\ell_1 \lor \ell_2 \cdots \lor \ell_{k-2} \lor u_{k-3}) \land (\ell_{k-1} \lor \ell_k \lor \neg u_{k-3})
  \]

21.1.2.5 An Example

Example 21.1.4 \( \phi = (\neg x_1 \lor \neg x_4) \land (x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_2 \lor \neg x_3 \lor x_4 \lor x_1) \land (x_1) \).

\[
\psi = (\neg x_1 \lor \neg x_4 \lor z) \land (\neg x_1 \lor \neg x_4 \lor \neg z) \land \\
(\neg x_2 \lor \neg x_3 \lor \neg x_4) \land (x_1 \lor \neg x_2 \lor \neg x_3) \land \\
(\neg x_2 \lor \neg x_3 \lor y_1) \land (x_4 \lor x_1 \lor \neg y_1) \land \\
(x_1 \lor u \lor v) \land (x_1 \lor u \lor \neg v) \land (x_1 \lor \neg u \lor v) \land (x_1 \lor \neg u \lor \neg v)
\]
21.1.2.6 Overall Reduction Algorithm

Input: CNF formula \( \varphi \)
for each clause \( c \) of \( \varphi \)
    if \( c \) does not have exactly 3 literals
        construct \( c' \) as before
    else
        \( c' = c \)
\( \psi \) is conjunction of all \( c' \) constructed in loop
is \( \psi \) satisfiable?

Correctness (informal)
\( \varphi \) is satisfiable iff \( \psi \) is satisfiable because for each clause \( c \), the new 3CNF formula \( c' \) is logically equivalent to \( c \).

21.1.2.7 What about 2-SAT?

2-SAT can be solved in polynomial time!
No known polynomial time reduction from SAT (or 3-SAT) to 2-SAT. If there was, then SAT and 3-SAT would be solvable in polynomial time.

21.1.3 3-SAT and Independent Set

21.1.3.1 3-SAT \( \leq_P \) Independent Set

Input Given a 3CNF formula \( \varphi \)

Goal Construct a graph \( G_\varphi \) and number \( k \) such that \( G_\varphi \) has an independent set of size \( k \)
iff \( \varphi \) is satisfiable. \( G_\varphi \) should be constructible in time polynomial in size of \( \varphi \)

Importance of reduction: Although 3-SAT is much more expressive, it can be reduced to a seemingly specialized Independent Set problem.

21.1.3.2 Interpreting 3-SAT

There are two ways to think about 3-SAT

- Find a way to assign 0/1 (false/true) to the variables such that the formula evaluates to true, that is each clause evaluates to true

- Pick a literal from each clause and find a truth assignment to make all of them true. You will fail if two of the literals you pick are in conflict, i.e., you pick \( x_i \) and \( \neg x_i \)

We will take the second view of 3-SAT to construct the reduction.
21.1.3.3 The Reduction

- $G_\varphi$ will have one vertex for each literal in a clause
- Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
- Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict
- Take $k$ to be the number of clauses

21.1.3.4 Correctness

**Proposition 21.1.5** $\varphi$ is satisfiable iff $G_\varphi$ has an independent set of size $k$ ($= \text{number of clauses in } \varphi$).

*Proof*: 

$\Rightarrow$ Let $a$ be the truth assignment satisfying $\varphi$

- Pick one of the vertices, corresponding to true literals under $a$, from each triangle. This is an independent set of the appropriate size

$\Leftarrow$ Let $S$ be an independent set of size $k$

- $S$ must contain exactly one vertex from each clause
- $S$ cannot contain vertices labeled by conflicting clauses
- Thus, it is possible to obtain a truth assignment that makes in the literals in $S$ true; such an assignment satisfies one literal in every clause

21.1.3.5 Correctness (contd)

**Proposition 21.1.6** $\varphi$ is satisfiable iff $G_\varphi$ has an independent set of size $k$ ($= \text{number of clauses in } \varphi$).

*Proof*:

$\Leftarrow$ Let $S$ be an independent set of size $k$

- $S$ must contain exactly one vertex from each clause
- $S$ cannot contain vertices labeled by conflicting clauses
- Thus, it is possible to obtain a truth assignment that makes in the literals in $S$ true; such an assignment satisfies one literal in every clause

Figure 21.1: Graph for $\varphi = (\neg x_1 \lor x_2 \lor x_3) \land (x_1 \lor \neg x_2 \lor x_3) \land (\neg x_1 \lor x_2 \lor x_4)$
21.1.3.6 Transitivity of Reductions

\[ X \leq P Y \text{ and } Y \leq P Z \implies X \leq P Z. \]

Note: \( X \leq P Y \) does not imply that \( Y \leq P X \) and hence it is very important to know the FROM and TO in a reduction.

To prove \( X \leq P Y \) you need to show a reduction FROM \( X \) TO \( Y \)

In other words show that an algorithm for \( Y \) implies an algorithm for \( X \).

21.2 Definition of NP

21.2.0.7 Recap . . .

Problems

- Independent Set
- Vertex Cover
- Set Cover
- SAT
- 3-SAT

Relationship

\[
3\text{-SAT} \leq_P \text{Independent Set} \leq_P \text{Vertex Cover} \leq_P \text{Set Cover} \\
3\text{-SAT} \leq_P \text{SAT} \leq_P 3\text{-SAT}
\]

21.3 Preliminaries

21.3.1 Problems and Algorithms

21.3.1.1 Problems and Algorithms: Formal Approach

Decision Problems

- \textit{Problem Instance:} Binary string \( s \), with size \(|s|\)
- \textit{Problem:} A set \( X \) of strings on which the answer should be “yes”; we call these YES instances of \( X \). Strings not in \( X \) are NO instances of \( X \).

\textbf{Definition 21.3.1} \quad \textit{A is an algorithm for problem } X \textit{ if } A(s) = "yes" \textit{ iff } s \in X

\textit{A is said to have a polynomial running time if there is a polynomial } p(\cdot) \textit{ such that for every string } s, \textit{ } A(s) \textit{ terminates in at most } O(p(|s|)) \textit{ steps}
21.3.1.2 Polynomial Time

Definition 21.3.2 Polynomial time (denoted \( P \)) is the class of all (decision) problems that have an algorithm that solves it in polynomial time

Example 21.3.3 \( \Delta \) Problems in \( P \) include

- Is there a shortest path from \( s \) to \( t \) of length \( \leq k \) in \( G \)?
- Is there a flow of value \( \geq k \) in network \( G \)?
- Is there an assignment to variables to satisfy given linear constraints?

21.3.1.3 Efficiency Hypothesis

A problem \( X \) has an efficient algorithm iff \( X \in P \), that is \( X \) has a polynomial time algorithm.

Justifications:

- robustness of definition to variations in machines
- a sound theoretical definition
- most known polynomial time algorithms for “natural” problems have small polynomial running times

21.3.1.4 Problems with no known polynomial time algorithms

Problems

- Independent Set
- Vertex Cover
- Set Cover
- SAT
- 3-SAT

There are of course undecidable problems (no algorithm at all!) but many problems that we want to solve are like above.

Question: What is common to above problems?
21.3.1.5 Efficient Checkability

Above problems share the following feature:

For any YES instance $I_X$ of $X$ there is a proof/certificate/solution that is of length $\text{poly}(|I_X|)$ such that given a proof one can efficiently check that $I_X$ is indeed a YES instance.

Examples:

- SAT formula $\varphi$: proof is a satisfying assignment
- Independent Set in graph $G$ and $k$: a subset $S$ of vertices

21.3.2 Certifiers/Verifiers

21.3.2.1 Certifiers

Definition 21.3.4 An algorithm $C(\cdot, \cdot)$ is a certifier for problem $X$ if for every $s \in X$ there is some string $t$ such that $C(s,t) = "yes"$, and conversely, if for some $s$ and $t$, $C(s,t) = "yes"$ then $s \in X$.

The string $t$ is called a certificate or proof for $s$.

Efficient Certifier

$C$ is an efficient certifier for problem $X$ if there is a polynomial $p(\cdot)$ such that for every string $s$, $s \in X$ iff there is a string $t$ with $|t| \leq p(|s|)$, $C(s,t) = "yes"$ and $C$ runs in polynomial time.

21.3.2.2 Example: Independent Set

- Problem: Does $G = (V, E)$ have an independent set of size $\geq k$?
  - Certificate: Set $S \subseteq V$
  - Certifier: Check $|S| \geq k$ and no pair of vertices in $S$ is connected by an edge

21.3.3 Examples

21.3.3.1 Example: Vertex Cover

- Problem: Does $G$ have a vertex cover of size $\leq k$?
  - Certificate: $S \subseteq V$
  - Certifier: Check $|S| \leq k$ and that for every edge at least one endpoint is in $S$
21.3.3.2 Example: SAT

- **Problem**: Does formula \( \varphi \) have a satisfying truth assignment?
  - **Certificate**: Assignment of 0/1 values to each variable
  - **Certifier**: Check each clause under \( a \) and say “yes” if all clauses are true

21.3.3.3 Example: Composites

- **Problem**: Is number \( s \) a composite?
  - **Certificate**: A factor \( t \leq s \) such that \( t \neq 1 \) and \( t \neq s \)
  - **Certifier**: Check that \( t \) divides \( s \) (Euclid’s algorithm)

21.4 NP

21.4.1 Definition

21.4.1.1 Nondeterministic Polynomial Time

**Definition 21.4.1** Nondeterministic Polynomial Time (denoted by \( \text{NP} \)) is the class of all problems that have efficient certifiers

**Example 21.4.2** \( \text{\textit{i2-\delta Independent Set, Vertex Cover, Set Cover, SAT, 3-SAT, Composites are all examples of problems in \( \text{NP} \)}} \)

21.4.1.2 Asymmetry in Definition of NP

Note that only YES instances have a short proof/certificate. NO instances need not have a short certificate.

Example: SAT formula \( \varphi \). No easy way to prove that \( \varphi \) is NOT satisfiable!

More on this and co-NP later on.

21.4.2 Intractibility

21.4.2.1 \( P \) versus \( \text{NP} \)

**Proposition 21.4.3** \( P \subseteq \text{NP} \)

For a problem in \( P \) no need for a certificate!

**Proof**: Consider problem \( X \in P \) with algorithm \( A \). Need to demonstrate that \( X \) has an efficient certifier
• Certifier $C$ on input $s, t$, runs $A(s)$ and returns the answer

• $C$ runs in polynomial time

• If $s \in X$ then for every $t$, $C(s, t) = "yes"$

• If $s \notin X$ then for every $t$, $C(s, t) = "no"$

21.4.2.2 Exponential Time

Definition 21.4.4 Exponential Time (denoted $EXP$) is the collection of all problems that have an algorithm which on input $s$ runs in exponential time, i.e., $O(2^{\text{poly}(|s|)})$

Example: $O(2^n)$, $O(2^{n \log n})$, $O(2^{n^2})$, ...

21.4.2.3 $NP$ versus $EXP$

Proposition 21.4.5 $NP \subseteq EXP$

Proof: Let $X \in NP$ with certifier $C$. Need to design an exponential time algorithm for $X$

• For every $t$, with $|t| \leq p(|s|)$ run $C(s, t)$; answer “yes” if any one of these calls returns “yes”

• The above algorithm correctly solves $X$ (exercise)

• Algorithm runs in $O(q(|s| + |p(s)|)2^{p(|s|)})$, where $q$ is the running time of $C$

21.4.2.4 Examples

• SAT: try all possible truth assignment to variables

• Independent set: try all possible subsets of vertices

• Vertex cover: try all possible subsets of vertices

21.4.2.5 Is $NP$ efficiently solvable?

We know $P \subseteq NP \subseteq EXP$

Big Question
Is there are problem in $NP$ that does not belong to $P$? Is $P = NP$?