Polynomial Time Reductions

Lecture 20

April 7, 2011

Part I

Introduction to Reductions

Reductions

A reduction from Problem **X** to Problem **Y** means (informally) that if we have an algorithm for Problem **Y**, we can use it to find an algorithm for Problem **X**.

Using Reductions

- We use reductions to find algorithms to solve problems.
- We also use reductions to show that we can't find algorithms for some problems. (We say that these problems are hard.)

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Example 1: Bipartite Matching and Flows

How do we solve the Bipartite Matching Problem?

Given a bipartite graph $\mathbf{G} = (\mathbf{U} \cup \mathbf{V}, \mathbf{E})$ and number $\mathbf{k}$, does $\mathbf{G}$ have a matching of size $\geq \mathbf{k}$?

Solution

Reduce it to Max-Flow. $\mathbf{G}$ has a matching of size $\geq \mathbf{k}$ iff there is a flow from $\mathbf{s}$ to $\mathbf{t}$ of value $\geq \mathbf{k}$.

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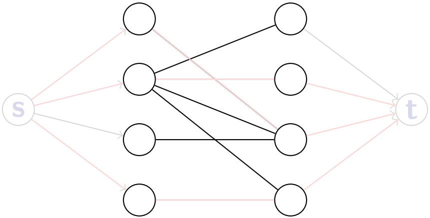

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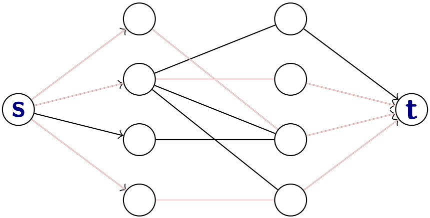

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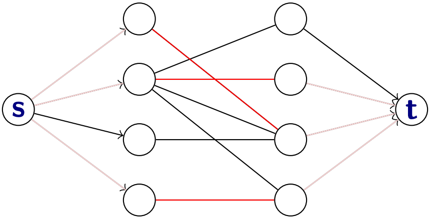

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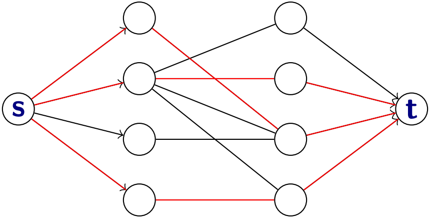

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Types of Problems

Decision, Search, and Optimization

- Decision problems (example: given n, is n prime?)
- Search problems (example: given n, find a factor of n if it exists)
- Optimization problems (example: find the smallest prime factor of n.)

For MAX-FLOW, the Optimization version is: Find the Maximum flow between s and t. The Decision Version is: Given an integer k, is there a flow of value $\geq k$ between s and t?

While using reductions and comparing problems, we typically work with the decision versions. Decision problems have Yes/No answers. This makes them easy to work with.

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Problems vs Instances

- A problem $\mathbf{\Pi}$ consists of an *infinite* collection of inputs $\{\mathbf{I}_1, \mathbf{I}_2, \ldots, \}$. Each input is referred to as an instance.
- The size of an instance $\mathbf{I}$ is the number of bits in its representation.
- For an instance $\mathbf{I}$, $\mathbf{sol(I)}$ is a set of feasible solutions to $\mathbf{I}$.
- For optimization problems each solution $\mathbf{s} \in \mathbf{sol(I)}$ has an associated value.

Examples

An instance of BIPARTITE MATCHING is a bipartite graph, and an integer $\mathbf{k}$. The solution to this instance is "YES" if the graph has a matching of size $\geq \mathbf{k}$, and "NO" otherwise.

An instance of MAX-FLOW is a graph $\mathbf{G}$ with edge-capacities, two vertices $\mathbf{s}, \mathbf{t}$, and an integer $\mathbf{k}$. The solution to this instance is "YES" if there is a flow from $\mathbf{s}$ to $\mathbf{t}$ of value $\geq \mathbf{k}$, else 'NO".

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Decision Problems and Languages

- A finite alphabet Σ. Σ^* is set of all finite strings on Σ.
- A language L is simply a subset of Σ^*; a set of strings.

For every language L there is an associated decision problem Π_L and conversely, for every decision problem Π there is an associated language L_Π.

- Given L, Π_L is the following problem: given $x \in \Sigma^*$, is $x \in L$? Each string in Σ^* is an instance of Π_L and L is the set of instances for which the answer is YES.
- Given Π the associated language $L_\Pi = \{I \mid I \text{ is an instance of } \Pi \text{ for which answer is YES}\}$.

Thus, decision problems and languages are used interchangeably.

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Reductions, revised.

For decision problems $\mathbf{X}, \mathbf{Y}$, a reduction from $\mathbf{X}$ to $\mathbf{Y}$ is:

- An algorithm ...
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(Actually, this is only one type of reduction, but this is the one we'll use most often.)

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Using reductions to solve problems

Given a reduction $\mathcal{R}$ from $\mathbf{X}$ to $\mathbf{Y}$, and an algorithm $\mathcal{A}_{\mathbf{Y}}$ for $\mathbf{Y}$:
We have an algorithm $\mathcal{A}_{\mathbf{X}}$ for $\mathbf{X}$! Here it is:

Given an instance $\mathbf{I}_{\mathbf{X}}$ of $\mathbf{X}$, use $\mathcal{R}$ to produce an instance $\mathbf{I}_{\mathbf{Y}}$ of $\mathbf{Y}$.
Now, use $\mathcal{A}_{\mathbf{Y}}$ to solve $\mathbf{I}_{\mathbf{Y}}$, and output the answer of $\mathcal{A}_{\mathbf{Y}}$.

In particular, if $\mathcal{R}$ and $\mathcal{A}_{\mathbf{Y}}$ are polynomial-time algorithms, $\mathcal{A}_{\mathbf{X}}$ is also polynomial-time.

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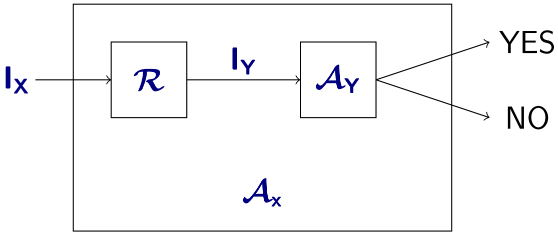

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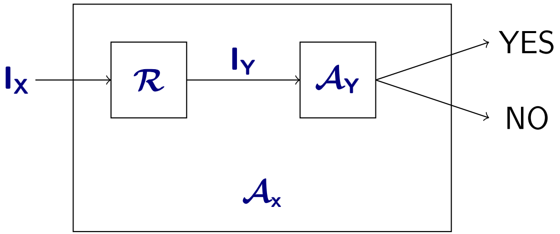

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Comparing Problems

- Reductions allow us to formalize the notion of "Problem **X** is no harder to solve than Problem **Y**".
- If Problem **X** reduces to Problem **Y** (we write $\mathbf{X} \leq \mathbf{Y}$), then **X** cannot be harder to solve than **Y**.
- Bipartite Matching $\leq$ Max-Flow. Therefore, Bipartite Matching cannot be harder than Max-Flow.
- Equivalently, Max-Flow is at least as hard as Bipartite Matching.
- More generally, if $\mathbf{X} \leq \mathbf{Y}$, we can say that **X** is no harder than **Y**, or **Y** is at least as hard as **X**.

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Part II

Examples of Reductions

Independent Sets and Cliques

Given a graph $\mathbf{G}$, a set of vertices $\mathbf{V}'$ is:

- An independent set if no two vertices of $\mathbf{V}'$ are connected by an edge of $\mathbf{G}$.
- A clique if *every* pair of vertices in $\mathbf{V}'$ is connected by an edge of $\mathbf{G}$.

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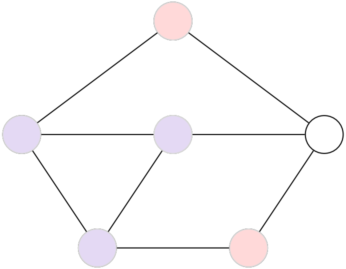

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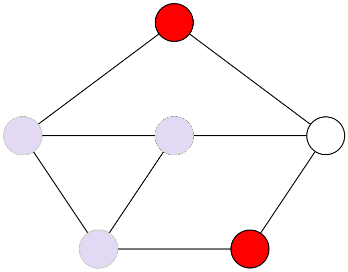

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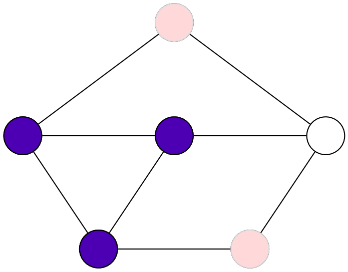

The Independent Set and Clique Problems

The Independent Set Problem:

Input A graph G and an integer k.

Goal Decide whether G has an independent set of size $\geq k$.

The Clique Problem:

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Reducing INDEPENDENT SET to CLIQUE

An instance of INDEPENDENT SET is a graph $\mathbf{G}$ and an integer $\mathbf{k}$.

Convert $\mathbf{G}$ to $\overline{\mathbf{G}}$, in which $(\mathbf{u}, \mathbf{v})$ is an edge iff $(\mathbf{u}, \mathbf{v})$ is not an edge of $\mathbf{G}$. ($\overline{\mathbf{G}}$ is the *complement* of $\mathbf{G}$.)
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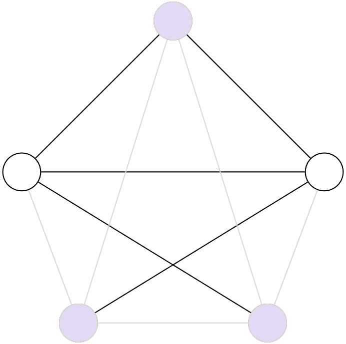

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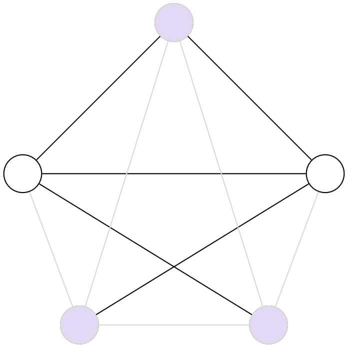

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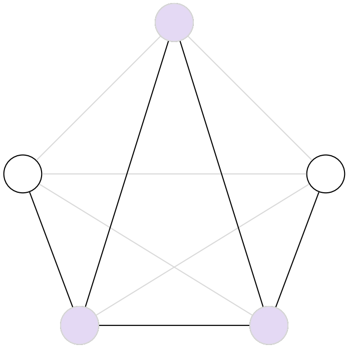

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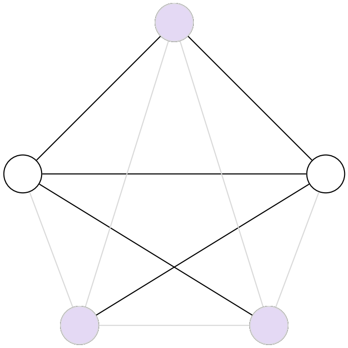

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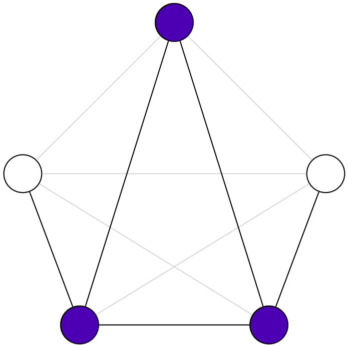

Independent Set and Clique

We showed that Independent Set $\leq$ Clique.
What does this mean?

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DFAs and NFAs

DFAs (Remember 273?) are automata that accept regular languages. NFAs are the same, except that they are non-deterministic, while DFAs are deterministic.

Every NFA can be converted to a DFA that accepts the same language using the subset construction.

(How long does this take?)
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DFA Universality

A DFA $\mathbf{M}$ is said to be universal if it accepts every string. That is, $\mathbf{L(M) = \Sigma^*}$, the set of all strings.

The DFA UNIVERSALITY Problem:

Input A DFA $\mathbf{M}$

Goal Decide whether $\mathbf{M}$ is universal.

How do we solve DFA UNIVERSALITY?
We check if $\mathbf{M}$ has *any* reachable non-final state.
Alternatively, minimize $\mathbf{M}$ to obtain $\mathbf{M'}$ and see if $\mathbf{M'}$ has a single state which is an accepting state.

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Reduce it to DFA UNIVERSALITY?

Given an NFA $\mathbf{N}$, convert it to an equivalent DFA $\mathbf{M}$, and use the DFA UNIVERSALITY Algorithm.

The reduction takes exponential time!

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Polynomial-time reductions

We say that an algorithm is efficient if it runs in polynomial-time.

To find efficient algorithms for problems, we are only interested in polynomial-time reductions. Reductions that take longer are not useful.

If we have a polynomial-time reduction from problem $\mathbf{X}$ to problem $\mathbf{Y}$ (we write $\mathbf{X} \leq_P \mathbf{Y}$), and a poly-time algorithm $\mathcal{A}_\mathbf{Y}$ for $\mathbf{Y}$, we have a polynomial-time/efficient algorithm for $\mathbf{X}$.

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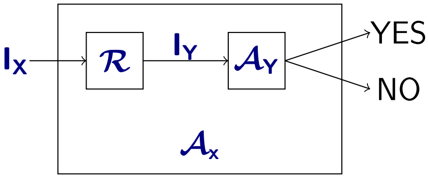

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A polynomial time reduction from a *decision* problem $\mathbf{X}$ to a *decision* problem $\mathbf{Y}$ is an *algorithm* $\mathcal{A}$ that has the following properties:

- given an instance $\mathbf{I_X}$ of $\mathbf{X}$, $\mathcal{A}$ produces an instance $\mathbf{I_Y}$ of $\mathbf{Y}$
- $\mathcal{A}$ runs in time polynomial in $|\mathbf{I_X}|$.
- Answer to $\mathbf{I_X}$ YES *iff* answer to $\mathbf{I_Y}$ is YES.

Proposition

If $\mathbf{X} \leq_{\mathbf{P}} \mathbf{Y}$ then a polynomial time algorithm for $\mathbf{Y}$ implies a polynomial time algorithm for $\mathbf{X}$.

Such a reduction is called a Karp reduction. Most reductions we will need are Karp reductions

Polynomial-time reductions and hardness

For decision problems $\mathbf{X}$ and $\mathbf{Y}$, if $\mathbf{X} \leq_{\mathbf{P}} \mathbf{Y}$, and $\mathbf{Y}$ has an efficient algorithm, $\mathbf{X}$ has an efficient algorithm.

If you believe that Independent Set does not have an efficient algorithm, why should you believe the same of Clique?

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If $\mathbf{X} \leq_{\mathbf{P}} \mathbf{Y}$ and $\mathbf{X}$ does not have an efficient algorithm, $\mathbf{Y}$ cannot have an efficient algorithm!

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Polynomial-time reductions and instance sizes

Proposition

Let $\mathcal{R}$ be a polynomial-time reduction from $\mathbf{X}$ to $\mathbf{Y}$. Then for any instance $\mathbf{I_X}$ of $\mathbf{X}$, the size of the instance $\mathbf{I_Y}$ of $\mathbf{Y}$ produced from $\mathbf{I_X}$ by $\mathcal{R}$ is polynomial in the size of $\mathbf{I_X}$.

Proof.

$\mathcal{R}$ is a polynomial-time algorithm and hence on input $\mathbf{I_X}$ of size $|\mathbf{I_X}|$ it runs in time $\mathbf{p}(|\mathbf{I_X}|)$ for some polynomial $\mathbf{p}()$.

$\mathbf{I_Y}$ is the output of $\mathcal{R}$ on input $\mathbf{I_X}$

$\mathcal{R}$ can write at most $\mathbf{p}(|\mathbf{I_X}|)$ bits and hence $|\mathbf{I_Y}| \leq \mathbf{p}(|\mathbf{I_X}|)$. □

Note: Converse is not true. A reduction need not be polynomial-time even if output of reduction is of size polynomial in its input.

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Transitivity of Reductions

Proposition

$\mathbf{X} \leq_{\mathbf{P}} \mathbf{Y}$ *and* $\mathbf{Y} \leq_{\mathbf{P}} \mathbf{Z}$ *implies that* $\mathbf{X} \leq_{\mathbf{P}} \mathbf{Z}$.

Note: $\mathbf{X} \leq_{\mathbf{P}} \mathbf{Y}$ does not imply that $\mathbf{Y} \leq_{\mathbf{P}} \mathbf{X}$ and hence it is very important to know the FROM and TO in a reduction.

To prove $\mathbf{X} \leq_{\mathbf{P}} \mathbf{Y}$ you need to show a reduction FROM $\mathbf{X}$ TO $\mathbf{Y}$
In other words show that an algorithm for $\mathbf{Y}$ implies an algorithm for $\mathbf{X}$.

Vertex Cover

Given a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, a set of vertices $\mathbf{S}$ is:

- A vertex cover if every $\mathbf{e} \in \mathbf{E}$ has at least one endpoint in $\mathbf{S}$.

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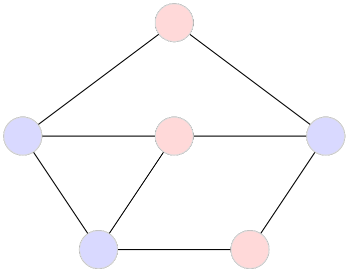

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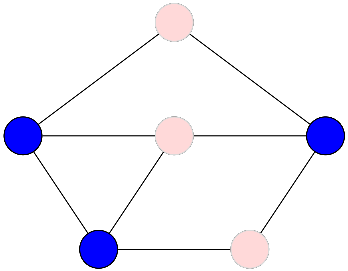

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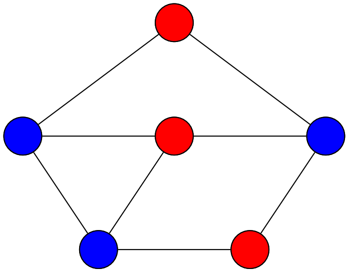

The VERTEX COVER Problem

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Input A graph $\mathbf{G}$ and integer $\mathbf{k}$

Goal Decide whether there is a vertex cover of size $\leq \mathbf{k}$

Can we relate INDEPENDENT SET and VERTEX COVER?

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Relationship between Vertex Cover and Independent Set

Proposition

Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ *be a graph.* $\mathbf{S}$ *is an independent set if and only if* $\mathbf{V} \setminus \mathbf{S}$ *is a vertex cover*

Proof.

($\Rightarrow$) Let $\mathbf{S}$ be an independent set

- Consider any edge $(\mathbf{u}, \mathbf{v}) \in \mathbf{E}$
- Since $\mathbf{S}$ is an independent set, either $\mathbf{u} \not\in \mathbf{S}$ or $\mathbf{v} \not\in \mathbf{S}$
- Thus, either $\mathbf{u} \in \mathbf{V} \setminus \mathbf{S}$ or $\mathbf{v} \in \mathbf{V} \setminus \mathbf{S}$
- $\mathbf{V} \setminus \mathbf{S}$ is a vertex cover

($\Leftarrow$) Let $\mathbf{V} \setminus \mathbf{S}$ be some vertex cover

- Consider $\mathbf{u}, \mathbf{v} \in \mathbf{S}$
- $(\mathbf{u}, \mathbf{v})$ is not edge, as otherwise $\mathbf{V} \setminus \mathbf{S}$ does not cover $(\mathbf{u}, \mathbf{v})$

Independent Set $\leq_P$ Vertex Cover

Let G, a graph with n vertices, and an integer k be an instance of the Independent Set problem.

G has an independent set of size $\geq k$ iff G has a vertex cover of size $\leq n - k$

(G, k) is an instance of Independent Set , and $(G, n - k)$ is an instance of Vertex Cover with the same answer.

Therefore, Independent Set $\leq_P$ Vertex Cover. Also Vertex Cover $\leq_P$ Independent Set.

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A problem of Languages

Suppose you work for the United Nations. Let **U** be the set of all languages spoken by people across the world. The United Nations also has a set of translators, all of whom speak English, and some other languages from **U**.

Due to budget cuts, you can only afford to keep **k** translators on your payroll. Can you do this, while still ensuring that there is someone who speaks every language in **U**?

More General problem: Find/Hire a small group of people who can accomplish a large number of tasks.

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The Set Cover Problem

Input Given a set $\mathbf{U}$ of $\mathbf{n}$ elements, a collection $\mathbf{S_1, S_2, \ldots S_m}$ of subsets of $\mathbf{U}$, and an integer $\mathbf{k}$

Goal Is there is a collection of at most $\mathbf{k}$ of these sets $\mathbf{S_i}$ whose union is equal to $\mathbf{U}$?

Example

Let $\mathbf{U} = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathbf{k} = 2$ with

$$
\begin{array}{ll}
S_1 = \{3,7\} & S_2 = \{3,4,5\} \\
S_3 = \{1\} & S_4 = \{2,4\} \\
S_5 = \{5\} & S_6 = \{1,2,6,7\}
\end{array}
$$

$\{S_2, S_6\}$ is a set cover

The Set Cover Problem

Input Given a set $\mathbf{U}$ of $\mathbf{n}$ elements, a collection $\mathbf{S_1, S_2, \ldots S_m}$ of subsets of $\mathbf{U}$, and an integer $\mathbf{k}$

Goal Is there is a collection of at most $\mathbf{k}$ of these sets $\mathbf{S_i}$ whose union is equal to $\mathbf{U}$?

Example

Let $\mathbf{U} = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathbf{k} = 2$ with

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\begin{array}{ll}
S_1 = \{3, 7\} & S_2 = \{3, 4, 5\} \\
S_3 = \{1\} & S_4 = \{2, 4\} \\
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\end{array}
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Vertex Cover $\leq_P$ Set Cover

Given graph $G = (V, E)$ and integer k as instance of Vertex Cover, construct an instance of Set Cover as follows:

- Number k for the Set Cover instance is the same as the number k given for the Vertex Cover instance.
- $U = E$
- We will have one set corresponding to each vertex; $S_v = \{e \mid e \text{ is incident on } v\}$

Observe that G has vertex cover of size k if and only if $U, \{S_v\}_{v \in V}$ has a set cover of size k. (Exercise: Prove this.)

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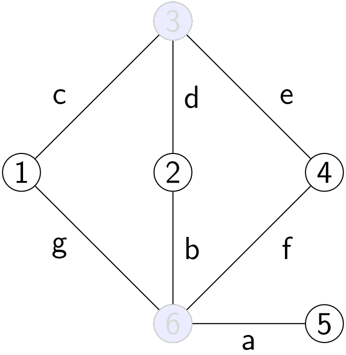
Vertex Cover $\leq_P$ Set Cover: Example



Let $U = \{a, b, c, d, e, f, g\}$, $k = 2$ with

$S_1 = \{c, g\}$	$S_2 = \{b, d\}$
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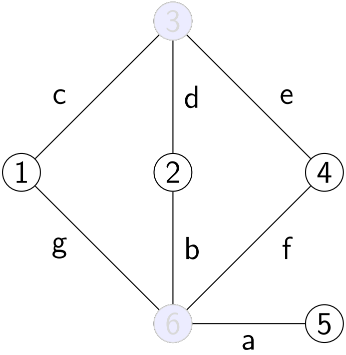

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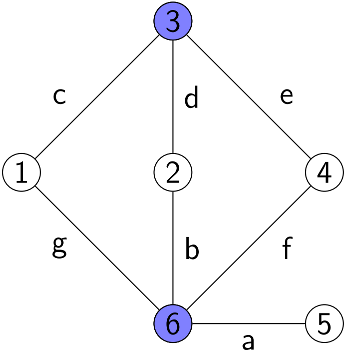

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Proving Reductions

To prove that $\mathbf{X} \leq_{\mathbf{P}} \mathbf{Y}$ you need to give an algorithm $\mathcal{A}$ that

- transforms an instance $\mathbf{I_X}$ of $\mathbf{X}$ into an instance $\mathbf{I_Y}$ of $\mathbf{Y}$
- satisfies the property that answer to $\mathbf{I_X}$ is YES iff $\mathbf{I_Y}$ is YES
  - typical easy direction to prove: answer to $\mathbf{I_Y}$ is YES if answer to $\mathbf{I_X}$ is YES
  - typical difficult direction to prove: answer to $\mathbf{I_X}$ is YES if answer to $\mathbf{I_Y}$ is YES (equivalenly answer to $\mathbf{I_X}$ is NO if answer to $\mathbf{I_Y}$ is NO)
- runs in *polynomial* time

Example of incorrect reduction proof

Try proving MATCHING $\leq_P$ BIPARTITE MATCHING via following reduction:

- Given graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ obtain a bipartite graph $\mathbf{G}' = (\mathbf{V}', \mathbf{E}')$ as follows.
  - Let $\mathbf{V}_1 = \{\mathbf{u}_1 \mid \mathbf{u} \in \mathbf{V}\}$ and $\mathbf{V}_2 = \{\mathbf{u}_2 \mid \mathbf{u} \in \mathbf{V}\}$. We set $\mathbf{V}' = \mathbf{V}_1 \cup \mathbf{V}_2$ (that is, we make two copies of $\mathbf{V}$)
  - $\mathbf{E}' = \{(\mathbf{u}_1, \mathbf{v}_2) \mid \mathbf{u} \neq \mathbf{v} \text{ and } (\mathbf{u}, \mathbf{v}) \in \mathbf{E}\}$
- Given $\mathbf{G}$ and integer $\mathbf{k}$ the reduction outputs $\mathbf{G}'$ and $\mathbf{k}$.

Example

"Proof"

Claim

Reduction is a poly-time algorithm. If $\mathbf{G}$ has a matching of size $\mathbf{k}$ then $\mathbf{G'}$ has a matching of size $\mathbf{k}$.

Proof.

Exercise. □

Claim

If $\mathbf{G'}$ has a matching of size $\mathbf{k}$ then $\mathbf{G}$ has a matching of size $\mathbf{k}$.

Incorrect! Why? Vertex $\mathbf{u} \in \mathbf{V}$ has two copies $\mathbf{u_1}$ and $\mathbf{u_2}$ in $\mathbf{G'}$. A matching in $\mathbf{G'}$ may use both copies!

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Summary

We looked at polynomial-time reductions.

Using polynomial-time reductions

- If $X \leq_P Y$, and we have an efficient algorithm for Y, we have an efficient algorithm for X.
- If $X \leq_P Y$, and there is no efficient algorithm for X, there is no efficient algorithm for Y.

We looked at some examples of reductions between INDEPENDENT SET, CLIQUE, VERTEX COVER, and SET COVER.

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