Chapter 18

Applications of Network Flows

CS 473: Fundamental Algorithms, Spring 2011
March 31, 2011

18.1 Important Properties of Flows

18.1.0.1 Network Flow: Facts to Remember

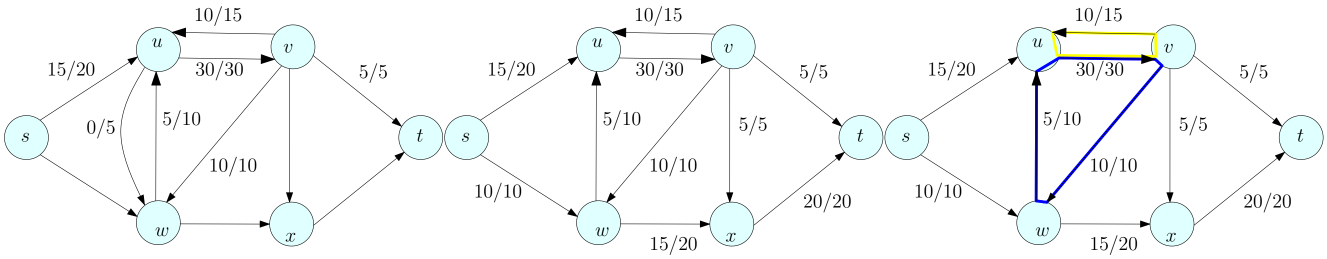
Flow network: directed graph G, capacities c, source s, sink t

- maximum s-t flow can be computed
  - using Ford-Fulkerson algorithm in $O(mC)$ time when capacities are integral and C is an upper bound on the flow
  - using variant of algorithm in homework in $O(m^2 \log C)$ time when capacities are integral
  - using Edmonds-Karp algorithm in $O(m^2 n)$ time when capacities are rational (strongly polynomial time algorithm)
- if capacities are integral then there is a maximum flow that is integral and above algorithms give an integral max flow
- given a flow of value v, can decompose into $O(m+n)$ flow paths of same total value v. integral flow implies integral flow on paths.
- maximum flow is equal to the minimum cut and minimum cut can be found in $O(m+n)$ time given any maximum flow

18.1.0.2 Paths, Cycles and Acyclicity of Flows

Definition 18.1.1 *Given a flow network $G = (V, E)$ and a flow $f : E \rightarrow \mathbb{R}^{\geq 0}$ on the edges, the support of f is the set of edges $E' \subseteq E$ with non-zero flow on them. That is, $E' = \{e \in E \mid f(e) > 0\}$.*

Question: Given flow f, can there by cycles in its support?



18.1.0.3 Acyclicity of Flows

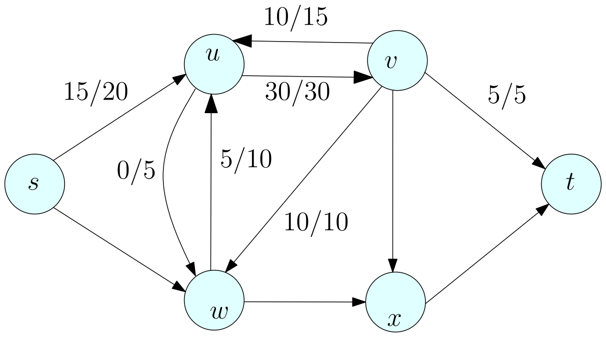
Proposition 18.1.2 *In any flow network, if f is a flow then there is another flow f' such that the support of f' is an* acyclic *graph and $v(f') = v(f)$. Further if f is an integral flow then so is f'.*

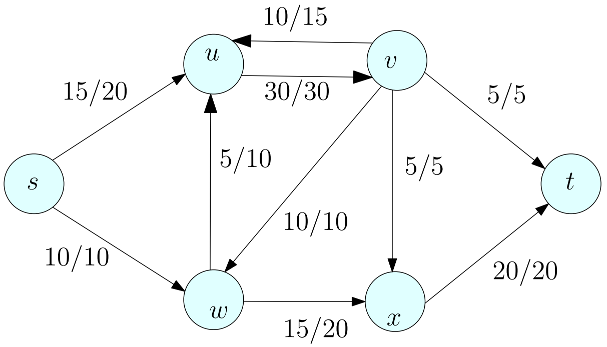
Proof:

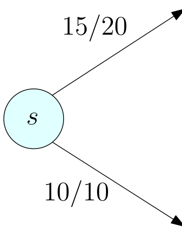
- $E' = \{e \in E \mid f(e) > 0\}$, support of f.
- Suppose there is a directed cycle C in E'
- Let e' be the edge in C with least amount of flow
- For each $e \in C$, reduce flow by $f(e')$. Remains a flow. Why?
- flow on e' is reduced to 0
- Claim: flow value from s to t does not change. Why?
- Iterate until no cycles

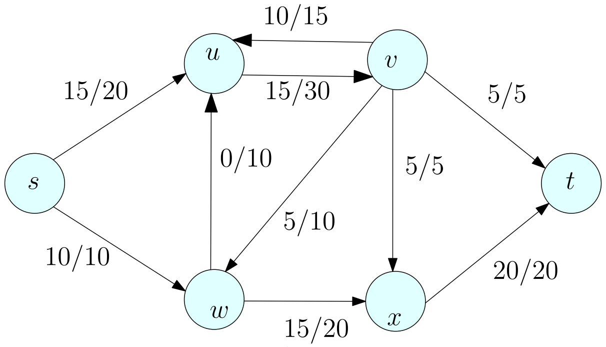
■

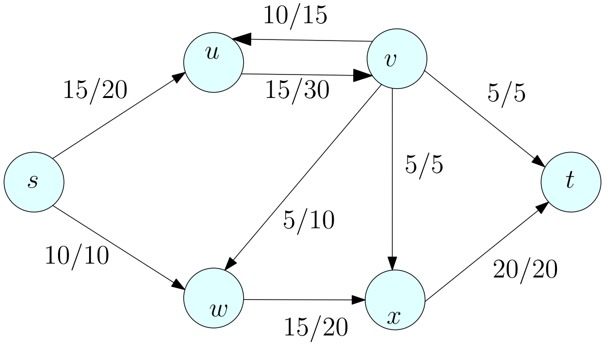
18.1.0.4 Example

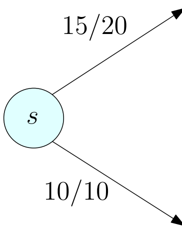


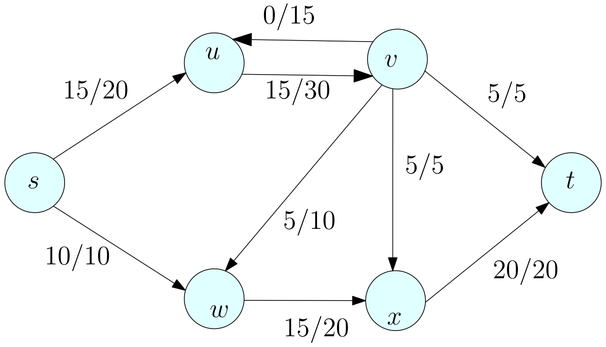


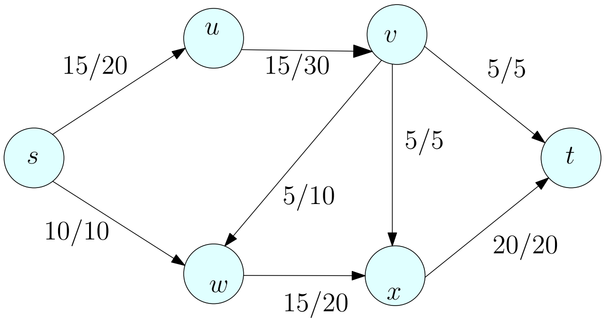


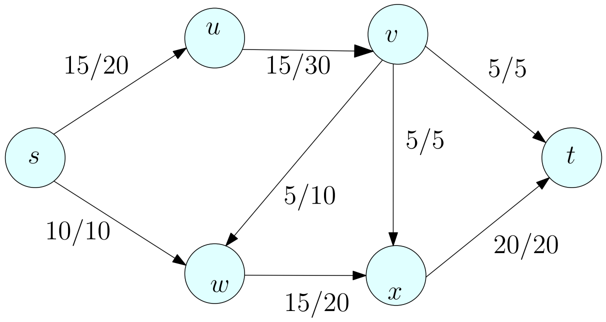




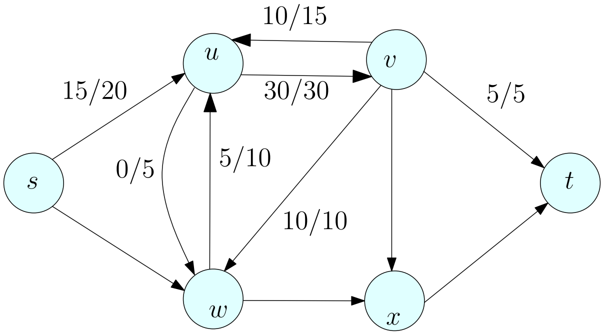








Throw away edge with no flow on itFind a cycle in the support/flowReduce flow on cycle as much as possibleThrow away edge with no flow on itFind a cycle in the support/flowReduce flow on cycle as much as possibleThrow away edge with no flow on itViola!!! An equivalent flow with no cycles in it. Original flow:



18.1.0.5 Flow Decomposition

Lemma 18.1.3 *Given an edge based flow $f : E \to \mathbb{R}^{\geq 0}$, there exists a collection of paths $\mathcal{P}$ and cycles $\mathcal{C}$ and an assignment of flow to them $f' : \mathcal{P} \cup \mathcal{C} \to \mathbb{R}^{\geq 0}$ such that:*

- $|\mathcal{P} \cup \mathcal{C}| \leq m$
- *for each $e \in E$, $\sum_{P \in \mathcal{P} : e \in P} f'(P) + \sum_{C \in \mathcal{C} : e \in C} f'(C) = f(e)$*
- $v(f) = \sum_{P \in \mathcal{P}} f'(P)$.
- *if f is integral then so are $f'(P)$ and $f'(C)$ for all P and C*

Proof:[Proof Idea]

- first remove all cycles as in previous proposition
- then decompose into paths as in previous lecture.
- Exercise: verify claims.

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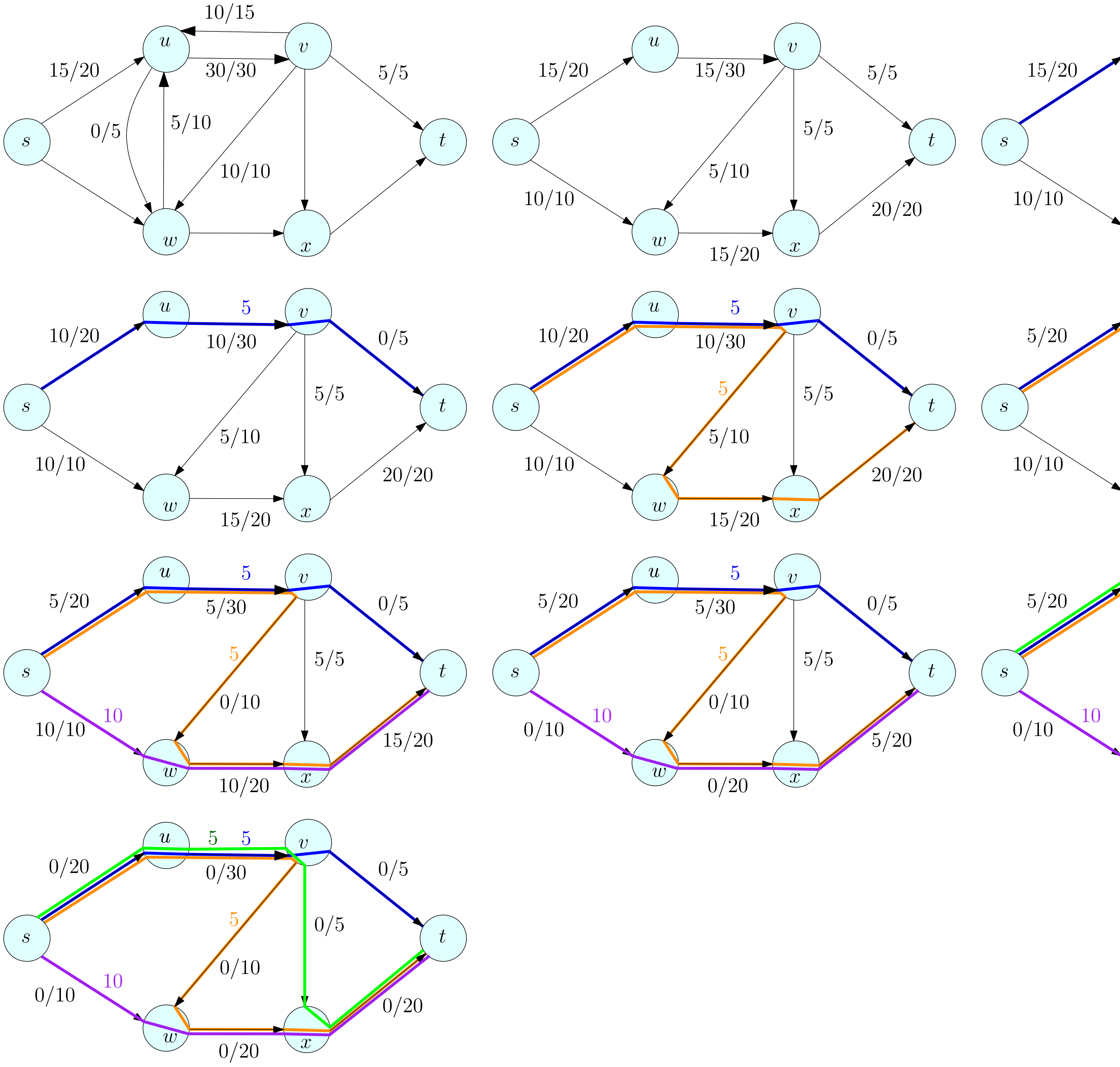
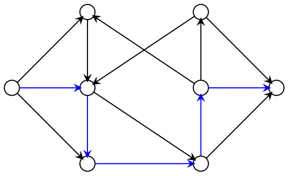
18.1.0.6 Example



Find cycles as shown beforeFind a source to sink path, and push max flow along it (5 unites)Compute remaining flowFind a source to sink path, and push max flow along it (5 unites). Edges with 0 flow on them can not be used as they are no longer in the support of the flow.Compute remaining flowFind a source to sink path, and push max flow along it (10 unites). Compute remaining flowFind a source to sink path, and push max flow along

Definition 18.3.1



it (5 unites). Compute remaining flowNo flow remains in the graph. We fully decomposed the flow into flow on paths. Together with the cycles, we get a decomposition of the original flow into m flows on paths and cycles.

18.1.0.7 Flow Decomposition

Lemma 18.1.4 *Given an edge based flow $f : E \to \mathbb{R}^{\geq 0}$, there exists a collection of paths $\mathcal{P}$ and cycles $\mathcal{C}$ and an assignment of flow to them $f' : \mathcal{P} \cup \mathcal{C} \to \mathbb{R}^{\geq 0}$ such that:*

- $|\mathcal{P} \cup \mathcal{C}| \leq m$
- *for each $e \in E$, $\sum_{P \in \mathcal{P}: e \in P} f'(P) + \sum_{C \in \mathcal{C}: e \in C} f'(C) = f(e)$*
- $v(f) = \sum_{P \in \mathcal{P}} f'(P)$.
- *if f is integral then so are $f'(P)$ and $f'(C)$ for all P and C*

Above flow decomposition can be computed in $O(m^2)$ time.

18.2 Network Flow Applications I

18.3 Edge Disjoint Paths

18.3.1 Directed Graphs

18.3.1.1 Edge-Disjoint Paths in Directed Graphs

A set of paths is edge disjoint if no two paths share an edge.

Problem

Given a directed graph with two special vertices s and t, find the *maximum* number of edge disjoint paths from s to t

Applications: Fault tolerance in routing — edges/nodes in networks can fail. Disjoint paths allow for planning backup routes in case of failures.

18.3.2 Reduction to Max-Flow

18.3.2.1 Reduction to Max-Flow

Problem
Given a directed graph G with two special vertices s and t, find the maximum number of edge disjoint paths from s to t

Reduction
Consider G as a flow network with edge capacities 1, and find max-flow.

18.3.2.2 Correctness of Reduction

Lemma 18.3.2 *If G has k edge disjoint paths $P_1, P_2, \ldots, P_k$ then there is an s-t flow of value k.*

Proof: Set $f(e) = 1$ if e belongs to one of the paths $P_1, P_2, \ldots, P_k$; other-wise set $f(e) = 0$. This defines a flow of value k. ■

18.3.2.3 Correctness of Reduction

Lemma 18.3.3 *If G has a flow of value k then there are k edge disjoint paths between s and t.*

Proof:

- Capacities are all 1 and hence there is integer flow of value k, that is $f(e) = 0$ or $f(e) = 1$ for each e.
- Decompose flow into paths of same value
- Flow on each path is either 1 or 0
- Hence there are k paths $P_1, P_2, \ldots, P_k$ with flow of 1 each
- Paths are edge-disjoint since capacities are 1.

■

18.3.2.4 Running Time

Theorem 18.3.4 *The number of edge disjoint paths in G can be found in $O(mn)$ time.*

Run Ford-Fulkerson algorithm. Maximum possible flow is n and hence run-time is $O(nm)$.

18.3.3 Menger's Theorem

18.3.3.1 Menger's Theorem

Theorem 18.3.5 (Menger) *Let G be a directed graph. The minimum number of edges whose removal disconnects s from t (the minimum-cut between s and t) is equal to the maximum number of edge-disjoint paths in G between s and t.*

Proof: Maxflow-mincut theorem and integrality of flow. ■

Menger proved his theorem before Maxflow-Mincut theorem! Maxflow-Mincut theorem is a generalization of Menger's theorem to capacitated graphs.

18.3.4 Undirected Graphs

18.3.4.1 Edge Disjoint Paths in Undirected Graphs

Problem
Given an *undirected* graph G, find the maximum number of edge disjoint paths in G

Reduction:

- create *directed* graph H by adding directed edges (u, v) and (v, u) for each edge uv in G.
- compute maximum s-t flow in H

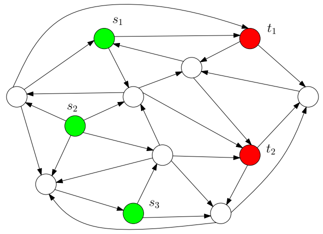
Problem: Both edges (u, v) and (v, u) may have non-zero flow!

Not a Problem! Can assume maximum flow in H is acyclic and hence cannot have non-zero flow on both (u, v) and (v, u). Reduction works. See book for more details.

18.4 Multiple Sources and Sinks

18.4.0.2 Multiple Sources and Sinks

- Directed graph G with edge capacities $c(e)$
- source nodes $s_1, s_2, \ldots, s_k$
- sink nodes $t_1, t_2, \ldots, t_\ell$
- sources and sinks are *disjoint*



18.4.0.3 Multiple Sources and Sinks

- Directed graph G with edge capacities $c(e)$
- source nodes $s_1, s_2, \ldots, s_k$
- sink nodes $t_1, t_2, \ldots, t_\ell$
- sources and sinks are *disjoint*

Maximum Flow: send as much flow as possible from the sources to the sinks. *Sinks don't care which source they get flow from.*

Minimum Cut: find a minimum capacity set of edge E' such that removing E' disconnects every source from every sink.

18.4.0.4 Multiple Sources and Sinks: Formal Definition

- Directed graph G with edge capacities $c(e)$
- source nodes $s_1, s_2, \ldots, s_k$
- sink nodes $t_1, t_2, \ldots, t_\ell$
- sources and sinks are *disjoint*

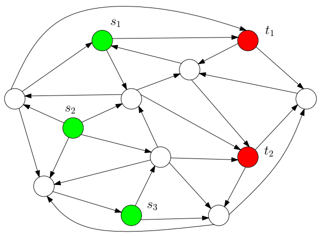
A function $f : E \to \mathbb{R}^{\geq 0}$ is a flow if:

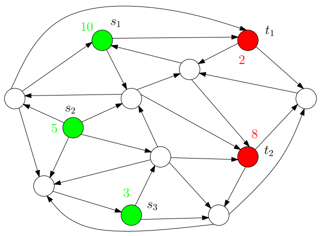
- for each $e \in E$, $f(e) \leq c(e)$ and
- for each v which is not a source or a sink $f^{\text{in}}(v) = f^{\text{out}}(v)$.

Goal: max $\sum_{i=1}^{k}(f^{\text{out}}(s_i) - f^{\text{in}}(s_i))$, that is, flow out of sources

18.4.0.5 Reduction to Single-Source Single-Sink

- Add a *source* node s and a *sink* node t
- Add edges $(s, s_1), (s, s_2), \ldots, (s, s_k)$
- Add edges $(t_1, t), (t_2, t), \ldots, (t_\ell, t)$
- Set the capacity of the new edges to be ∞





18.4.0.6 Supplies and Demands

A further generalization:

- source s_i has a supply of $S_i \geq 0$
- since t_j has a demand of $D_j \geq 0$ units

Question: is there a flow from source to sinks such that supplies are not exceeded and demands are met? Formally we have the additional constraints that $f^{\text{out}}(s_i) - f^{\text{in}}(s_i) \leq S_i$ for each source s_i and $f^{\text{in}}(t_j) - f^{\text{out}}(t_j) \leq D_j$ for each sink t_j.

18.5 Bipartite Matching

18.5.1 Definitions

18.5.1.1 Matching

Input Given a (undirected) graph $G = (V, E)$

Goal Find a matching of maximum cardinality

- A matching is $M \subseteq E$ such that at most one edge in M is incident on any vertex

18.5.1.2 Bipartite Matching

Input Given a bipartite graph $G = (L \cup R, E)$

Goal Find a matching of maximum cardinality

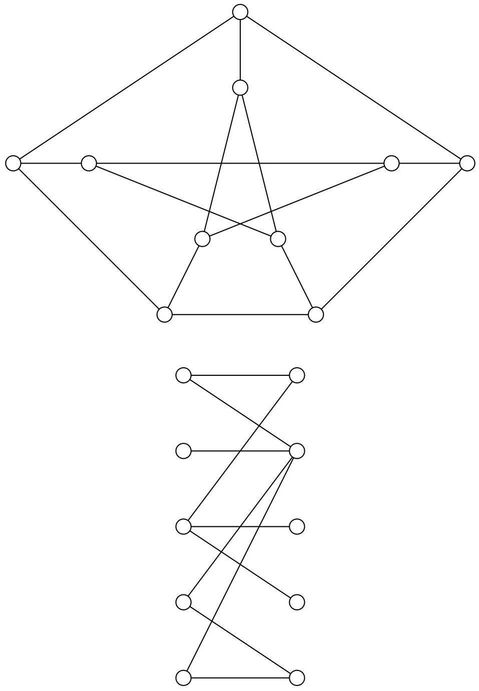


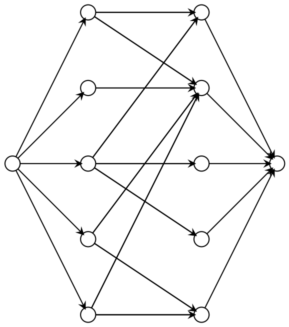
Figure 18.1: Maximum matching has 4 edges

18.5.2 Reduction to Max-Flow

18.5.2.1 Reduction to Max-Flow

Max-Flow Construction
Given graph $G = (L \cup R, E)$ create flow-network $G' = (V', E')$ as follows:

- $V' = L \cup R \cup \{s, t\}$ where s and t are the new source and sink
- Direct all edges in E from L to R, and add edges from s to all vertices in L and from each vertex in R to t
- Capacity of every edge is 1



18.5.2.2 Correctness: Matching to Flow

Proposition 18.5.1 *If G has a matching of size k then G' has a flow of value k.*

Proof: Let M be matching of size k. Let $M = \{(u_1, v_1), \ldots, (u_k, v_k)\}$. Consider following flow f in G':

- $f(s, u_i) = 1$ and $f(v_i, t) = 1$ for $1 \leq i \leq k$
- $f(u_i, v_i) = 1$ for $1 \leq i \leq k$
- for all other edges flow is zero.

Verify that f is a flow of value k (because M is a matching). ■

18.5.2.3 Correctness: Flow to Matching

Proposition 18.5.2 *If G' has a flow of value k then G has a matching of size k.*

Proof: Consider flow f of value k.

- Can assume f is integral. Thus each edge has flow 1 or 0
- Consider the set M of edges from L to R that have flow 1
  - M has k edges because value of flow is equal to the number of non-zero flow edges crossing cut $(L \cup \{s\}, R \cup \{t\})$
  - Each vertex has at most one edge in M incident upon it. Why?

■

18.5.2.4 Correctness of Reduction

Theorem 18.5.3 *The maximum flow value in G' = maximum cardinality of matching in G*

Consequence
Thus, to find maximum cardinality matching in G, we construct G' and find the maximum flow in G'. Note that the matching itself (not just the value) can be found efficiently from the flow.

18.5.2.5 Running Time

For graph G with n vertices and m edges G' has $O(n+m)$ edges, and $O(n)$ vertices.

- Generic Ford-Fulkerson: Running time is $O(mC) = O(nm)$ since $C = n$
- Capacity scaling: Running time is $O(m^2 \log C) = O(m^2 \log n)$

Better known running time: $O(m\sqrt{n})$

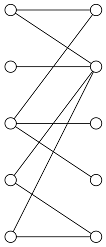


Figure 18.2: This graph does not have a perfect matching

18.5.3 Perfect Matchings

18.5.3.1 Perfect Matchings

Definition 18.5.4 *A matching M is said to be perfect if every vertex has one edge in M incident upon it.*

18.5.3.2 Characterizing Perfect Matchings

Problem
When does a bipartite graph have a perfect matching?

- Clearly $|L| = |R|$
- Are there any necessary and sufficient conditions?

18.5.3.3 A Necessary Condition

Lemma 18.5.5 *If $G = (L \cup R, E)$ has a perfect matching then for any $X \subseteq L$, $|N(X)| \geq |X|$, where $N(X)$ is the set of neighbors of vertices in X*

Proof: Since G has a perfect matching, every vertex of X is matched to a different neighbor, and so $|N(X)| \geq |X|$ ■

18.5.3.4 Hall's Theorem

Theorem 18.5.6 (Frobenius-Hall) *Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. G has a perfect matching if and only if for every $X \subseteq L$, $|N(X)| \geq |X|$*

One direction is the necessary condition.

For the other direction we will show the following:

- create flow network G' from G
- if $|N(X)| \geq |X|$ for all X, show that minimum s-t cut in G' is of capacity $n = |L| = |R|$
- implies that G has a perfect matching

18.5.3.5 Proof of Sufficiency

Assume $|N(X)| \geq |X|$ for each $X \in L$. Then show that min s-t cut in G' is of capacity n.

Let (A, B) be an *arbitrary* s-t cut in G'

- let $X = A \cap L$ and $Y = A \cap R$
- cut capacity is at least $(|L| - |X|) + |Y| + |N(X) - Y|$
- $|N(X) - Y| \geq |N(X)| - |Y|$ and by assumption $|N(X)| \geq |X|$ and hence $|N(X) - Y| \geq |X| - |Y|$
- cut capacity is therefore at least $|L| - |X| + |Y| + |X| - |Y| \geq |L| = n$.

18.5.3.6 Application: assigning jobs to people

- n jobs or tasks
- m people
- for each job a set of people who can do that job
- for each person j a limit on number of jobs k_j
- *Goal:* find an assignment of jobs to people so that all jobs are assigned and no person is overloaded

Reduce to max-flow similar to matching. Arises in many settings. Using *minimum-cost flows* can also handle the case when assigning a job i to person j costs c_{ij} and goal is assign all jobs but minimize cost of assignment.

18.5.3.7 Reduction to Maximum Flow

- Create directed graph $G = (V, E)$ as follows
  - $V = \{s, t\} \cup L \cup R$: L set of n jobs, R set of m people
  - add edges (s, i) for each job $i \in L$, capacity 1
  - add edges (j, t) for each person $j \in R$, capacity k_j
  - if job i can be done by person j add an edge (i, j), capacity 1
- Compute max s-t flow. There is an assignment if and only if flow value is n.