Applications of Network Flows

Lecture 18

March 31, 2011

Network Flow: Facts to Remember

Flow network: directed graph G, capacities c, source s, sink t

- maximum s-t flow can be computed
  - using Ford-Fulkerson algorithm in $O(mC)$ time when capacities are integral and C is an upper bound on the flow
  - using variant of algorithm in homework in $O(m^2 \log C)$ time when capacities are integral
  - using Edmonds-Karp algorithm in $O(m^2 n)$ time when capacities are rational (strongly polynomial time algorithm)
- if capacities are integral then there is a maximum flow that is integral and above algorithms give an integral max flow
- given a flow of value v, can decompose into $O(m + n)$ flow paths of same total value v. integral flow implies integral flow on paths.
- maximum flow is equal to the minimum cut and minimum cut can be found in $O(m + n)$ time given any maximum flow

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Paths, Cycles and Acyclicity of Flows

Definition

Given a flow network $G = (V, E)$ and a flow $f : E \to \mathbb{R}^{\geq 0}$ on the edges, the support of f is the set of edges $E' \subseteq E$ with non-zero flow on them. That is, $E' = \{e \in E \mid f(e) > 0\}$.

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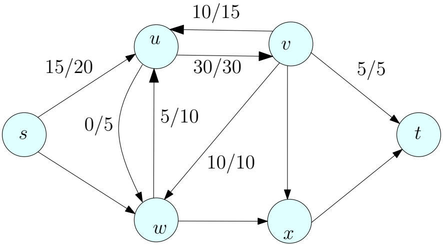

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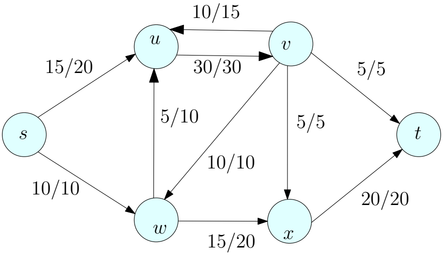

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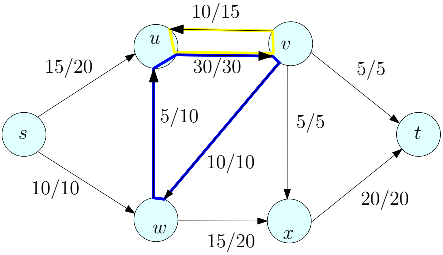


Acyclicity of Flows

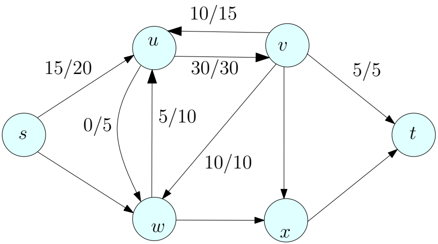
Proposition

In any flow network, if $\mathbf{f}$ *is a flow then there is another flow* $\mathbf{f'}$ *such that the support of* $\mathbf{f'}$ *is an* acyclic *graph and* $\mathbf{v(f') = v(f)}$*. Further if* $\mathbf{f}$ *is an integral flow then so is* $\mathbf{f'}$*.*

Proof.

- $\mathbf{E' = \{e \in E \mid f(e) > 0\}}$, support of $\mathbf{f}$.
- Suppose there is a directed cycle $\mathbf{C}$ in $\mathbf{E'}$
- Let $\mathbf{e'}$ be the edge in $\mathbf{C}$ with least amount of flow
- For each $\mathbf{e \in C}$, reduce flow by $\mathbf{f(e')}$. Remains a flow. Why?
- flow on $\mathbf{e'}$ is reduced to $\mathbf{0}$
- Claim: flow value from $\mathbf{s}$ to $\mathbf{t}$ does not change. Why?
- Iterate until no cycles □

Example



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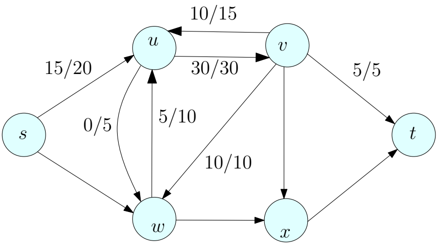
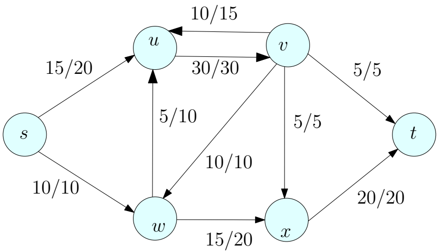
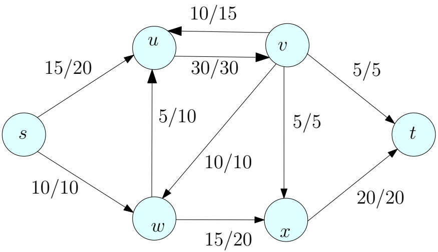
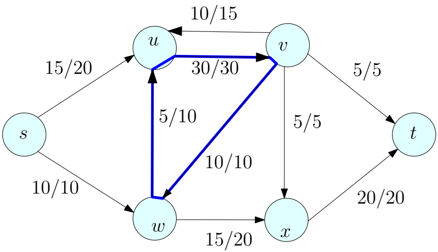




Throw away edge with no flow on it

Example





Find a cycle in the support/flow

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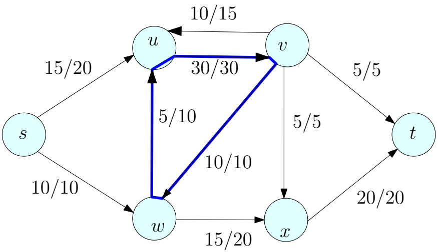
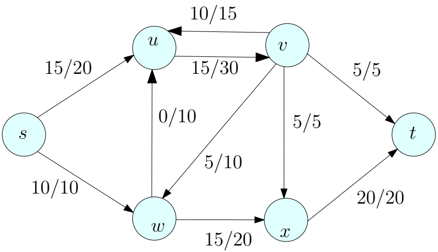
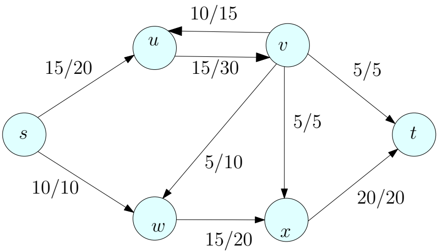


Reduce flow on cycle as much as possible

Example





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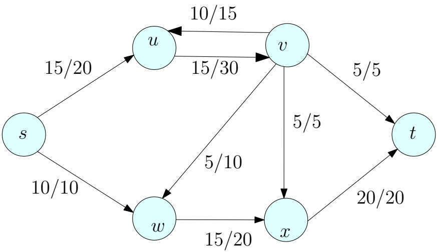

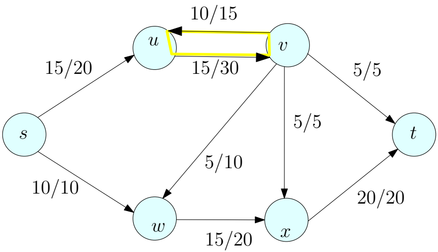

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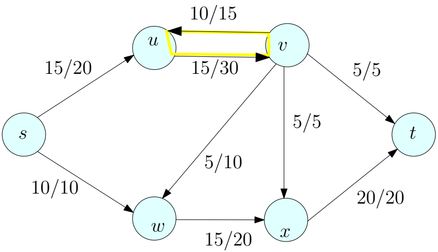

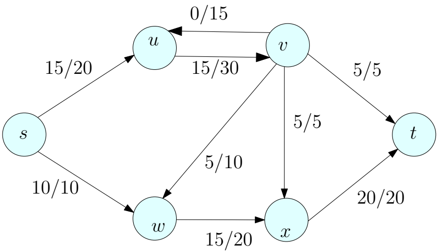

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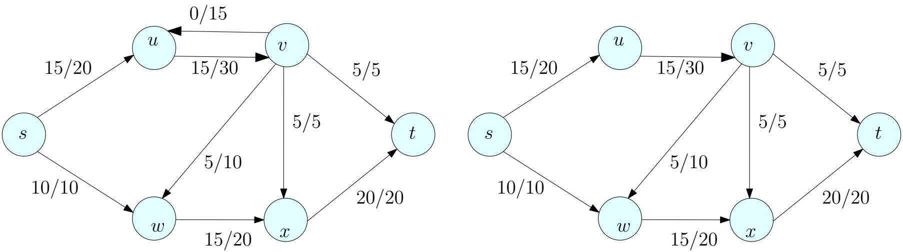

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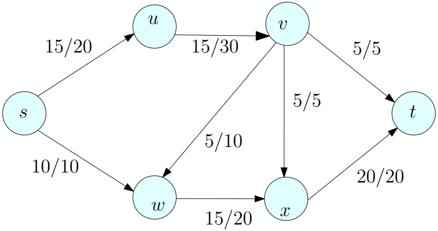
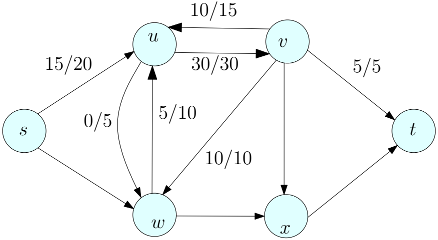


Viola!!! An equivalent flow with no cycles in it. Original flow:



Flow Decomposition

Lemma

Given an edge based flow $\mathbf{f} : \mathbf{E} \to \mathbb{R}^{\geq 0}$, *there exists a collection of paths* $\mathcal{P}$ *and cycles* $\mathcal{C}$ *and an assignment of flow to them* $\mathbf{f}' : \mathcal{P} \cup \mathcal{C} \to \mathbb{R}^{\geq 0}$ *such that:*

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Proof Idea.

- first remove all cycles as in previous proposition
- then decompose into paths as in previous lecture.
- Exercise: verify claims.

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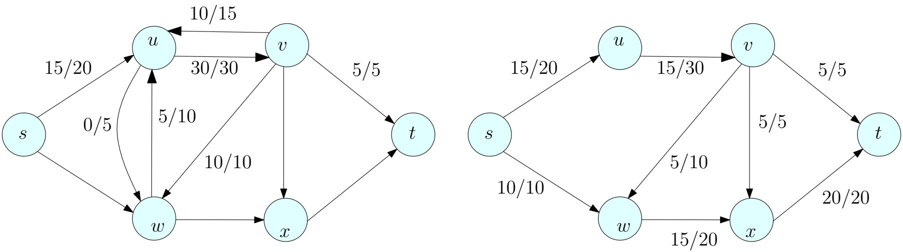

Find cycles as shown before

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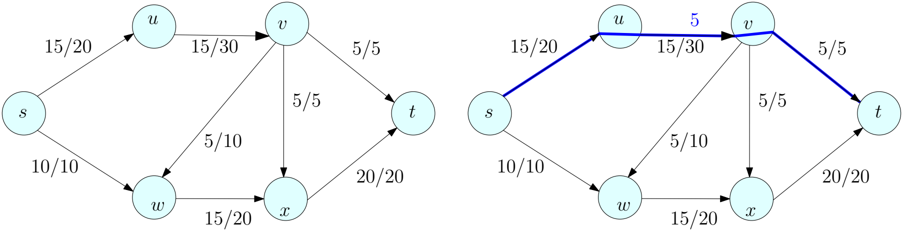

Find a source to sink path, and push max flow along it (5 unites)

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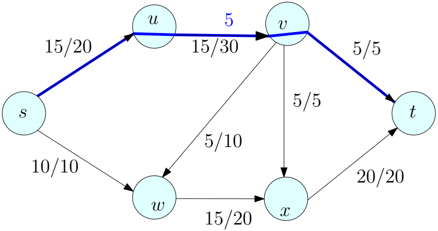

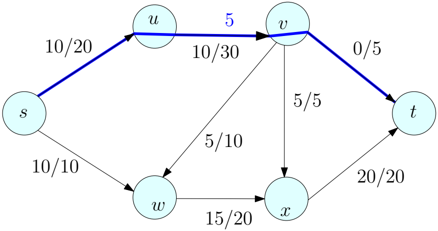

Compute remaining flow

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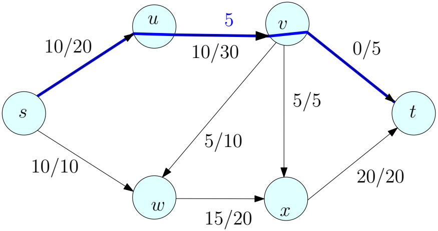

Find a source to sink path, and push max flow along it (5 unites). Edges with **0** flow on them can not be used as they are no longer in the support of the flow.

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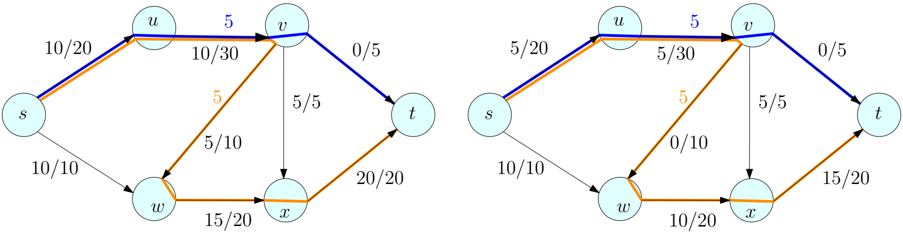

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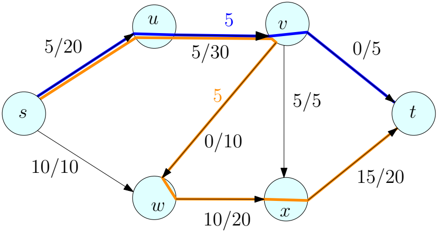

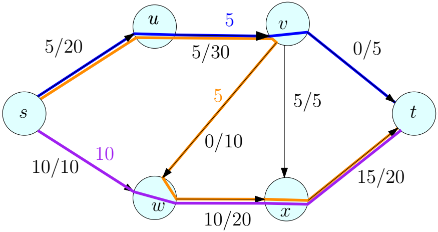

Find a source to sink path, and push max flow along it (10 unites).

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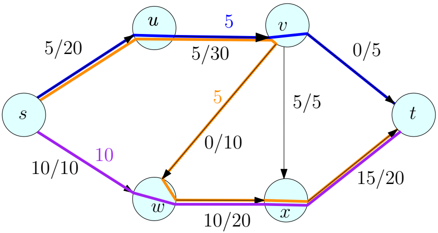

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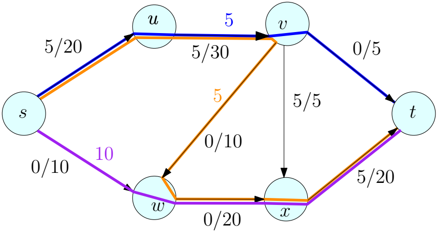

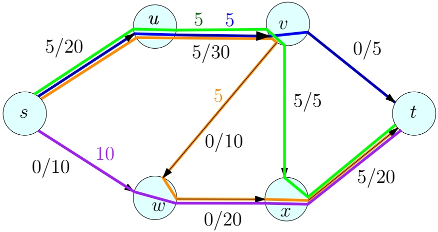

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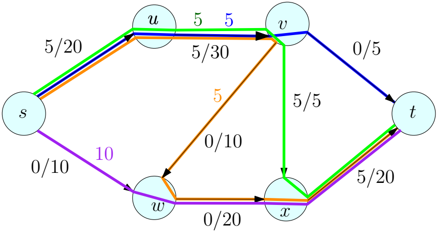

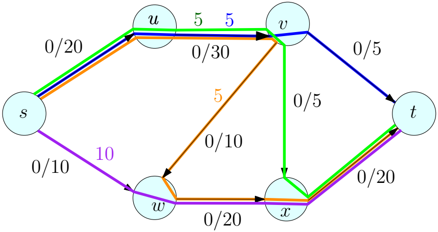

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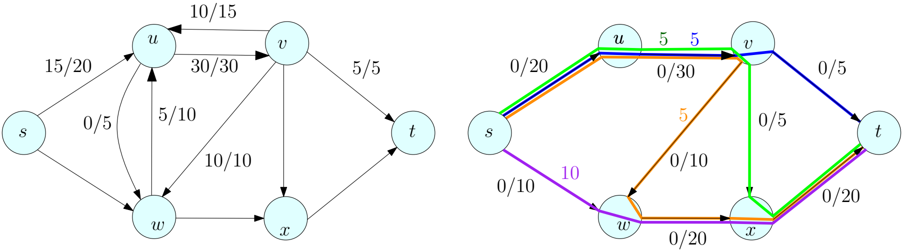

No flow remains in the graph. We fully decomposed the flow into flow on paths. Together with the cycles, we get a decomposition of the original flow into **m** flows on paths and cycles.

Flow Decomposition

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Above flow decomposition can be computed in $\mathbf{O}(\mathbf{m}^2)$ *time.*

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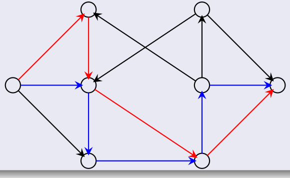
Part I

Network Flow Applications I

Edge-Disjoint Paths in Directed Graphs

Definition



A set of paths is edge disjoint if no two paths share an edge.

Problem

Given a directed graph with two special vertices **s** and **t**, find the *maximum* number of edge disjoint paths from **s** to **t**

Applications: Fault tolerance in routing — edges/nodes in networks can fail. Disjoint paths allow for planning backup routes in case of failures.

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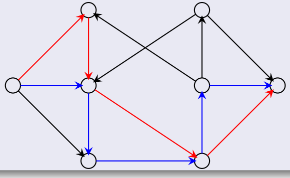

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Reduction to Max-Flow

Problem

Given a directed graph **G** with two special vertices **s** and **t**, find the maximum number of edge disjoint paths from **s** to **t**

Reduction

Consider **G** as a flow network with edge capacities 1, and find max-flow.

Correctness of Reduction

Lemma

If $\mathbf{G}$ *has* $\mathbf{k}$ *edge disjoint paths* $\mathbf{P_1, P_2, \ldots, P_k}$ *then there is an* $\mathbf{s}$-$\mathbf{t}$ *flow of value* $\mathbf{k}$.

Proof.

Set $\mathbf{f(e) = 1}$ if $\mathbf{e}$ belongs to one of the paths $\mathbf{P_1, P_2, \ldots, P_k}$; other-wise set $\mathbf{f(e) = 0}$. This defines a flow of value $\mathbf{k}$. □

Correctness of Reduction

Lemma

If $\mathbf{G}$ *has* $\mathbf{k}$ *edge disjoint paths* $\mathbf{P_1, P_2, \ldots, P_k}$ *then there is an* $\mathbf{s}$-$\mathbf{t}$ *flow of value* $\mathbf{k}$.

Proof.

Set $\mathbf{f(e) = 1}$ if $\mathbf{e}$ belongs to one of the paths $\mathbf{P_1, P_2, \ldots, P_k}$; other-wise set $\mathbf{f(e) = 0}$. This defines a flow of value $\mathbf{k}$. □

Correctness of Reduction

Lemma

If $\mathbf{G}$ has a flow of value $\mathbf{k}$ then there are $\mathbf{k}$ edge disjoint paths between $\mathbf{s}$ and $\mathbf{t}$.

Proof.

- Capacities are all $\mathbf{1}$ and hence there is integer flow of value $\mathbf{k}$, that is $\mathbf{f(e) = 0}$ or $\mathbf{f(e) = 1}$ for each $\mathbf{e}$.
- Decompose flow into paths of same value
- Flow on each path is either $\mathbf{1}$ or $\mathbf{0}$
- Hence there are $\mathbf{k}$ paths $\mathbf{P_1, P_2, \ldots, P_k}$ with flow of $\mathbf{1}$ each
- Paths are edge-disjoint since capacities are $\mathbf{1}$.

□

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If $\mathbf{G}$ *has a flow of value* $\mathbf{k}$ *then there are* $\mathbf{k}$ *edge disjoint paths between* $\mathbf{s}$ *and* $\mathbf{t}$*.*

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- Paths are edge-disjoint since capacities are $\mathbf{1}$.

□

Running Time

Theorem

The number of edge disjoint paths in $\mathbf{G}$ *can be found in* $\mathbf{O(mn)}$ *time.*

Run Ford-Fulkerson algorithm. Maximum possible flow is $\mathbf{n}$ and hence run-time is $\mathbf{O(nm)}$.

Menger's Theorem

Theorem (Menger)

Let **G** *be a directed graph. The minimum number of edges whose removal disconnects* **s** *from* **t** *(the minimum-cut between* **s** *and* **t***) is equal to the maximum number of edge-disjoint paths in* **G** *between* **s** *and* **t***.*

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Maxflow-mincut theorem and integrality of flow. □

Menger proved his theorem before Maxflow-Mincut theorem!
Maxflow-Mincut theorem is a generalization of Menger's theorem to capacitated graphs.

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Edge Disjoint Paths in Undirected Graphs

Problem

Given an undirected graph $\mathbf{G}$, find the maximum number of edge disjoint paths in $\mathbf{G}$

Reduction:

- create directed graph $\mathbf{H}$ by adding directed edges $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{v}, \mathbf{u})$ for each edge $\mathbf{uv}$ in $\mathbf{G}$.
- compute maximum $\mathbf{s}$-$\mathbf{t}$ flow in $\mathbf{H}$

Problem: Both edges $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{v}, \mathbf{u})$ may have non-zero flow!

Not a Problem! Can assume maximum flow in $\mathbf{H}$ is acyclic and hence cannot have non-zero flow on both $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{v}, \mathbf{u})$. Reduction works. See book for more details.

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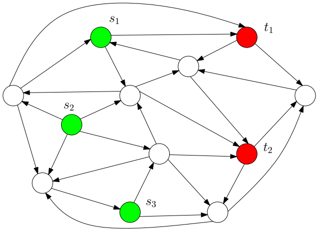
Multiple Sources and Sinks

- Directed graph $\mathbf{G}$ with edge capacities $\mathbf{c(e)}$
- source nodes $\mathbf{s_1, s_2, \ldots, s_k}$
- sink nodes $\mathbf{t_1, t_2, \ldots, t_\ell}$
- sources and sinks are *disjoint*



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Maximum Flow: send as much flow as possible from the sources to the sinks. *Sinks don't care which source they get flow from.*

Minimum Cut: find a minimum capacity set of edge $\mathbf{E'}$ such that removing $\mathbf{E'}$ disconnects every source from every sink.

Multiple Sources and Sinks: Formal Definition

- Directed graph $\mathbf{G}$ with edge capacities $\mathbf{c(e)}$
- source nodes $\mathbf{s_1, s_2, \ldots, s_k}$
- sink nodes $\mathbf{t_1, t_2, \ldots, t_\ell}$
- sources and sinks are *disjoint*

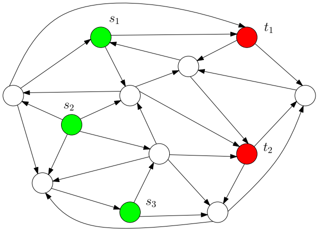
A function $\mathbf{f : E \rightarrow \mathbb{R}^{\geq 0}}$ is a flow if:

- for each $\mathbf{e \in E}$, $\mathbf{f(e) \leq c(e)}$ and
- for each $\mathbf{v}$ which is not a source or a sink $\mathbf{f^{in}(v) = f^{out}(v)}$.

Goal: max $\sum_{i=1}^{k}(\mathbf{f^{out}(s_i) - f^{in}(s_i)})$, that is, flow out of sources

Reduction to Single-Source Single-Sink

- Add a *source* node $\mathbf{s}$ and a *sink* node $\mathbf{t}$
- Add edges $(\mathbf{s}, \mathbf{s_1}), (\mathbf{s}, \mathbf{s_2}), \ldots, (\mathbf{s}, \mathbf{s_k})$
- Add edges $(\mathbf{t_1}, \mathbf{t}), (\mathbf{t_2}, \mathbf{t}), \ldots, (\mathbf{t_\ell}, \mathbf{t})$
- Set the capacity of the new edges to be ∞

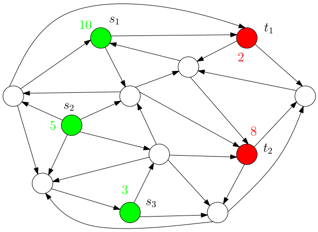


Supplies and Demands

A further generalization:

- source $\mathbf{s_i}$ has a supply of $\mathbf{S_i} \geq \mathbf{0}$
- since $\mathbf{t_j}$ has a demand of $\mathbf{D_j} \geq \mathbf{0}$ units

Question: is there a flow from source to sinks such that supplies are not exceeded and demands are met? Formally we have the additional constraints that $\mathbf{f^{out}(s_i) - f^{in}(s_i) \leq S_i}$ for each source $\mathbf{s_i}$ and $\mathbf{f^{in}(t_j) - f^{out}(t_j) \leq D_j}$ for each sink $\mathbf{t_j}$.



Matching

Input Given a (undirected) graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$

Goal Find a matching of maximum cardinality

- A matching is $\mathbf{M} \subseteq \mathbf{E}$ such that at most one edge in $\mathbf{M}$ is incident on any vertex

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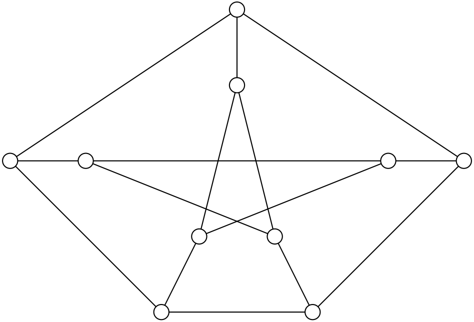

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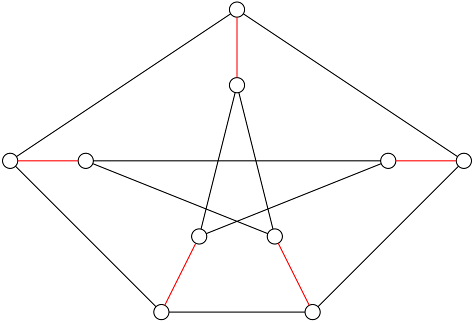

Bipartite Matching

Input Given a bipartite graph $\mathbf{G} = (\mathbf{L} \cup \mathbf{R}, \mathbf{E})$

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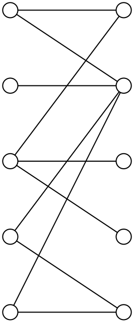

Figure: Maximum matching has 4 edges

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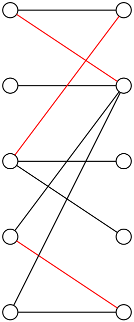

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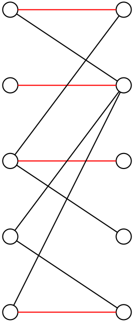

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Reduction to Max-Flow

Max-Flow Construction

Given graph $\mathbf{G} = (\mathbf{L} \cup \mathbf{R}, \mathbf{E})$ create flow-network $\mathbf{G}' = (\mathbf{V}', \mathbf{E}')$ as follows:

- $\mathbf{V}' = \mathbf{L} \cup \mathbf{R} \cup \{\mathbf{s}, \mathbf{t}\}$ where $\mathbf{s}$ and $\mathbf{t}$ are the new source and sink
- Direct all edges in $\mathbf{E}$ from $\mathbf{L}$ to $\mathbf{R}$, and add edges from $\mathbf{s}$ to all vertices in $\mathbf{L}$ and from each vertex in $\mathbf{R}$ to $\mathbf{t}$
- Capacity of every edge is $\mathbf{1}$

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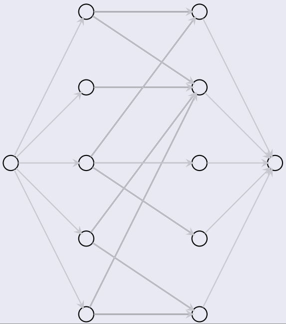

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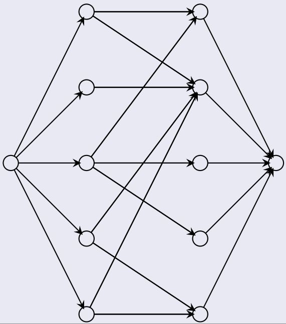

- $\mathbf{V' = L \cup R \cup \{s, t\}}$ where $\mathbf{s}$ and $\mathbf{t}$ are the new source and sink
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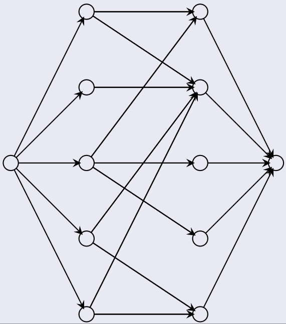

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- Capacity of every edge is $\mathbf{1}$

Correctness: Matching to Flow

Proposition

If $\mathbf{G}$ *has a matching of size* $\mathbf{k}$ *then* $\mathbf{G'}$ *has a flow of value* $\mathbf{k}$*.*

Proof.

Let $\mathbf{M}$ be matching of size $\mathbf{k}$. Let $\mathbf{M} = \{(u_1, v_1), \ldots, (u_k, v_k)\}$. Consider following flow $\mathbf{f}$ in $\mathbf{G'}$:

- $f(s, u_i) = 1$ and $f(v_i, t) = 1$ for $1 \le i \le k$
- $f(u_i, v_i) = 1$ for $1 \le i \le k$
- for all other edges flow is zero.

Verify that $\mathbf{f}$ is a flow of value $\mathbf{k}$ (because $\mathbf{M}$ is a matching). □

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Correctness: Flow to Matching

Proposition

If $\mathbf{G'}$ has a flow of value $\mathbf{k}$ then $\mathbf{G}$ has a matching of size $\mathbf{k}$.

Proof.

Consider flow $\mathbf{f}$ of value $\mathbf{k}$.

- Can assume $\mathbf{f}$ is integral. Thus each edge has flow $\mathbf{1}$ or $\mathbf{0}$
- Consider the set $\mathbf{M}$ of edges from $\mathbf{L}$ to $\mathbf{R}$ that have flow 1
  - $\mathbf{M}$ has $\mathbf{k}$ edges because value of flow is equal to the number of non-zero flow edges crossing cut $(\mathbf{L} \cup \{\mathbf{s}\}, \mathbf{R} \cup \{\mathbf{t}\})$
  - Each vertex has at most one edge in $\mathbf{M}$ incident upon it. Why?

□

Correctness of Reduction

Theorem

The maximum flow value in $\mathbf{G}'$ = *maximum cardinality of matching in* $\mathbf{G}$

Consequence

Thus, to find maximum cardinality matching in $\mathbf{G}$, we construct $\mathbf{G}'$ and find the maximum flow in $\mathbf{G}'$. Note that the matching itself (not just the value) can be found efficiently from the flow.

Running Time

For graph $\mathbf{G}$ with $\mathbf{n}$ vertices and $\mathbf{m}$ edges $\mathbf{G'}$ has $\mathbf{O(n+m)}$ edges, and $\mathbf{O(n)}$ vertices.

- Generic Ford-Fulkerson: Running time is $\mathbf{O(mC) = O(nm)}$ since $\mathbf{C = n}$
- Capacity scaling: Running time is $\mathbf{O(m^2 \log C) = O(m^2 \log n)}$

Better known running time: $\mathbf{O(m\sqrt{n})}$

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Perfect Matchings

Definition

A matching **M** is said to be perfect if every vertex has one edge in **M** incident upon it.

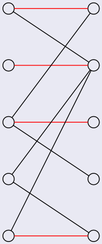

Figure: This graph does not have a perfect matching

Characterizing Perfect Matchings

Problem

When does a bipartite graph have a perfect matching?

- Clearly $|\mathbf{L}| = |\mathbf{R}|$
- Are there any necessary and sufficient conditions?

A Necessary Condition

Lemma

If $\mathbf{G} = (\mathbf{L} \cup \mathbf{R}, \mathbf{E})$ *has a perfect matching then for any* $\mathbf{X} \subseteq \mathbf{L}$*,* $|\mathbf{N}(\mathbf{X})| \geq |\mathbf{X}|$*, where* $\mathbf{N}(\mathbf{X})$ *is the set of neighbors of vertices in* $\mathbf{X}$

Proof.

Since $\mathbf{G}$ has a perfect matching, every vertex of $\mathbf{X}$ is matched to a different neighbor, and so $|\mathbf{N}(\mathbf{X})| \geq |\mathbf{X}|$ □

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Hall's Theorem

Theorem (Frobenius-Hall)

Let $\mathbf{G} = (\mathbf{L} \cup \mathbf{R}, \mathbf{E})$ be a bipartite graph with $|\mathbf{L}| = |\mathbf{R}|$. $\mathbf{G}$ has a perfect matching if and only if for every $\mathbf{X} \subseteq \mathbf{L}$, $|\mathbf{N}(\mathbf{X})| \geq |\mathbf{X}|$

One direction is the necessary condition.

For the other direction we will show the following:

- create flow network $\mathbf{G}'$ from $\mathbf{G}$
- if $|\mathbf{N}(\mathbf{X})| \geq |\mathbf{X}|$ for all $\mathbf{X}$, show that minimum $\mathbf{s}$-$\mathbf{t}$ cut in $\mathbf{G}'$ is of capacity $\mathbf{n} = |\mathbf{L}| = |\mathbf{R}|$
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Proof of Sufficiency

Assume $|\mathbf{N(X)}| \geq |\mathbf{X}|$ for each $\mathbf{X} \in \mathbf{L}$. Then show that min $\mathbf{s}$-$\mathbf{t}$ cut in $\mathbf{G'}$ is of capacity $\mathbf{n}$.

Let $(\mathbf{A}, \mathbf{B})$ be an *arbitrary* $\mathbf{s}$-$\mathbf{t}$ cut in $\mathbf{G'}$

- let $\mathbf{X} = \mathbf{A} \cap \mathbf{L}$ and $\mathbf{Y} = \mathbf{A} \cap \mathbf{R}$
- cut capacity is at least $(|\mathbf{L}| - |\mathbf{X}|) + |\mathbf{Y}| + |\mathbf{N(X)} - \mathbf{Y}|$
- $|\mathbf{N(X)} - \mathbf{Y}| \geq |\mathbf{N(X)}| - |\mathbf{Y}|$ and by assumption $|\mathbf{N(X)}| \geq |\mathbf{X}|$ and hence $|\mathbf{N(X)} - \mathbf{Y}| \geq |\mathbf{X}| - |\mathbf{Y}|$
- cut capacity is therefore at least $|\mathbf{L}| - |\mathbf{X}| + |\mathbf{Y}| + |\mathbf{X}| - |\mathbf{Y}| \geq |\mathbf{L}| = \mathbf{n}$.

Proof of Sufficiency

Assume $|N(X)| \geq |X|$ for each $X \in L$. Then show that min s-t cut in G' is of capacity n.

Let (A, B) be an *arbitrary* s-t cut in G'

- let $X = A \cap L$ and $Y = A \cap R$
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Let $(\mathbf{A}, \mathbf{B})$ be an *arbitrary* $\mathbf{s}$-$\mathbf{t}$ cut in $\mathbf{G}'$

- let $\mathbf{X} = \mathbf{A} \cap \mathbf{L}$ and $\mathbf{Y} = \mathbf{A} \cap \mathbf{R}$
- cut capacity is at least $(|\mathbf{L}| - |\mathbf{X}|) + |\mathbf{Y}| + |\mathbf{N}(\mathbf{X}) - \mathbf{Y}|$
- $|\mathbf{N}(\mathbf{X}) - \mathbf{Y}| \geq |\mathbf{N}(\mathbf{X})| - |\mathbf{Y}|$ and by assumption $|\mathbf{N}(\mathbf{X})| \geq |\mathbf{X}|$ and hence $|\mathbf{N}(\mathbf{X}) - \mathbf{Y}| \geq |\mathbf{X}| - |\mathbf{Y}|$
- cut capacity is therefore at least $|\mathbf{L}| - |\mathbf{X}| + |\mathbf{Y}| + |\mathbf{X}| - |\mathbf{Y}| \geq |\mathbf{L}| = \mathbf{n}$.

Application: assigning jobs to people

- n jobs or tasks
- m people
- for each job a set of people who can do that job
- for each person j a limit on number of jobs k_j
- Goal: find an assignment of jobs to people so that all jobs are assigned and no person is overloaded

Reduce to max-flow similar to matching.

Arises in many settings. Using *minimum-cost flows* can also handle the case when assigning a job i to person j costs c_{ij} and goal is assign all jobs but minimize cost of assignment.

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Reduction to Maximum Flow

- Create directed graph $\mathbf{G = (V, E)}$ as follows
  - $\mathbf{V = \{s, t\} \cup L \cup R}$: $\mathbf{L}$ set of $\mathbf{n}$ jobs, $\mathbf{R}$ set of $\mathbf{m}$ people
  - add edges $\mathbf{(s, i)}$ for each job $\mathbf{i \in L}$, capacity $\mathbf{1}$
  - add edges $\mathbf{(j, t)}$ for each person $\mathbf{j \in R}$, capacity $\mathbf{k_j}$
  - if job $\mathbf{i}$ can be done by person $\mathbf{j}$ add an edge $\mathbf{(i, j)}$, capacity $\mathbf{1}$
- Compute max $\mathbf{s}$-$\mathbf{t}$ flow. There is an assignment if and only if flow value is $\mathbf{n}$.

Matchings in General Graphs

Matchings in general graphs more complicated.

There is a polynomial time algorithm to compute a maximum matching in a general graph. Best known running time is $\mathbf{O(m\sqrt{n})}$.

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