Chapter 17

Network Flow Algorithms

CS 473: Fundamental Algorithms, Spring 2011
March 29, 2011

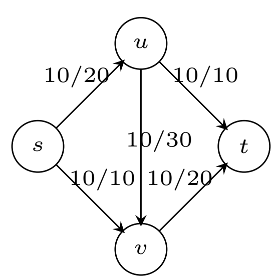
17.1 Algorithm(s) for Maximum Flow

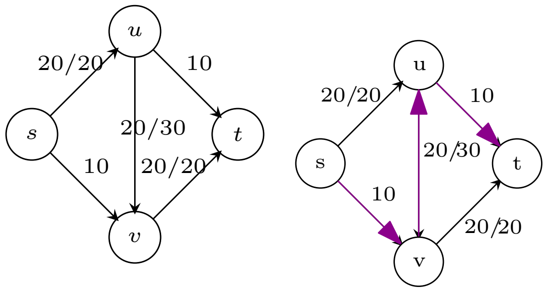
17.1.0.1 Greedy Approach

1. Begin with $f(e) = 0$ for each edge
2. Find a s-t path P with $f(e) < c(e)$ for every edge $e \in P$
3. *Augment* flow along this path
4. Repeat augmentation for as long as possible.

17.1.0.2 Greedy Approach: Issues

1. Begin with $f(e) = 0$ for each edge
2. Find a s-t path P with $f(e) < c(e)$ for every edge $e \in P$
3. Augment flow along this path





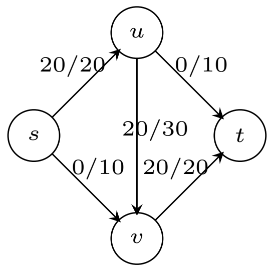


Figure 17.1: Flow in red edges

4. Repeat augmentation for as long as possible.

Greedy can get stuck in sub-optimal flow!
Need to "push-back" flow along edge (u, v)

17.2 Ford-Fulkerson Algorithm

17.2.0.3 Residual Graph

Definition 17.2.1 *For a network $G = (V, E)$ and flow f, the residual graph $G_f = (V', E')$ of G with respect to f is*

- $V' = V$
- *Forward Edges: For each edge $e \in E$ with $f(e) < c(e)$, we $e \in E'$ with capacity $c(e) - f(e)$*
- *Backward Edges: For each edge $e = (u, v) \in E$ with $f(e) > 0$, we $(v, u) \in E'$ with capacity $f(e)$*

17.2.0.4 Residual Graph Example

17.2.0.5 Residual Graph Property

Observation: Residual graph captures the "residual" problem exactly.

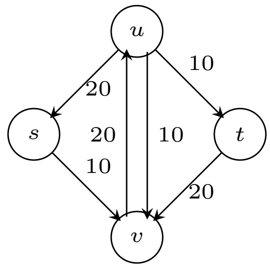


Figure 17.2: Residual Graph

Lemma 17.2.2 *Let f be a flow in G and G_f be the residual graph. If f' is a flow in G_f then $f + f'$ is a flow in G of value $v(f) + v(f')$.*

Lemma 17.2.3 *Let f and f' be two flows in G with $v(f') \geq v(f)$. Then there is a flow f'' of value $v(f') - v(f)$ in G_f.*

Definition of + and - for flows is intuitive and the above lemmas are easy in some sense but a bit messy to formally prove.

17.2.0.6 Residual Graph Property: Implication

Recursive algorithm for finding a maximum flow:

```
MaxFlow(G,s,t):
    If the flow from s to t is 0
        return 0
    Find any flow f with v(f) > 0 i
    Recursively compute a maximum f
    Output the flow f + f'
```

Iterative algorithm for finding a maximum flow:

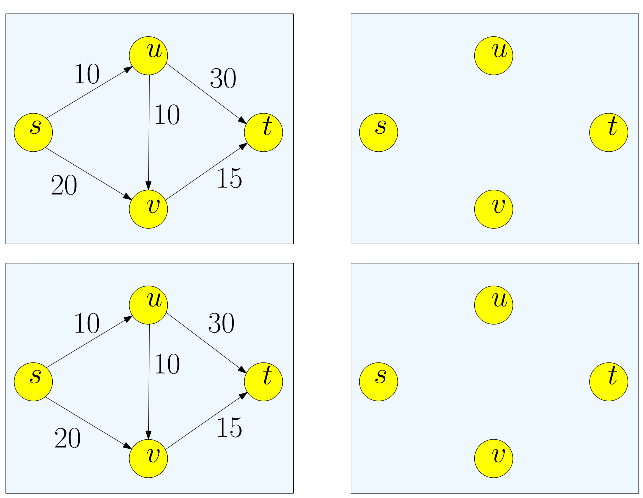
```
MaxFlow(G,s,t):
    Start with flow f that is 0 on
    While there is a flow f' in G_f
        f = f + f'
        Update G_f
    endWhile
    Output f
```

17.2.0.7 Ford-Fulkerson Algorithm

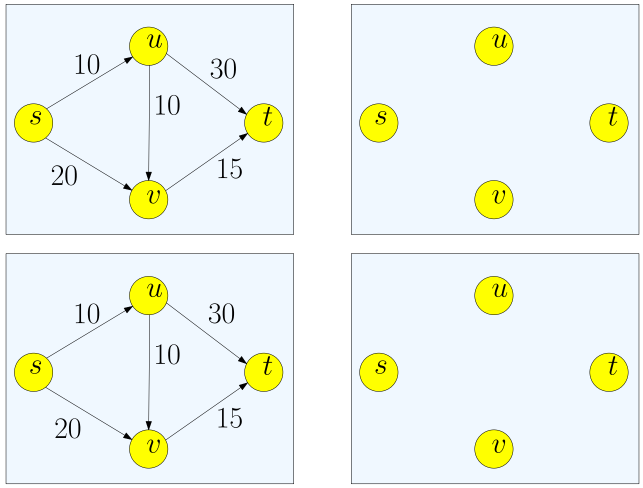
```
algFordFulkerson
    for every edge e, f(e) = 0
    G_f is residual graph of G with respect to f
    while G_f has a simple s-t path do
        let P be simple s-t path in G_f
        f = augment(f, P)
        Construct new residual graph G_f
```

```
augment(f,P)
    let b be bottleneck capacity,
        i.e., min capacity of edges in P (in G_f)
    for each edge (u,v) in P do
        if e = (u,v) is a forward edge then
            f(e) = f(e) + b
        else (* (u,v) is a backward edge *)
            let e = (v,u) (* (v,u) is in G *)
            f(e) = f(e) - b
    return f
```

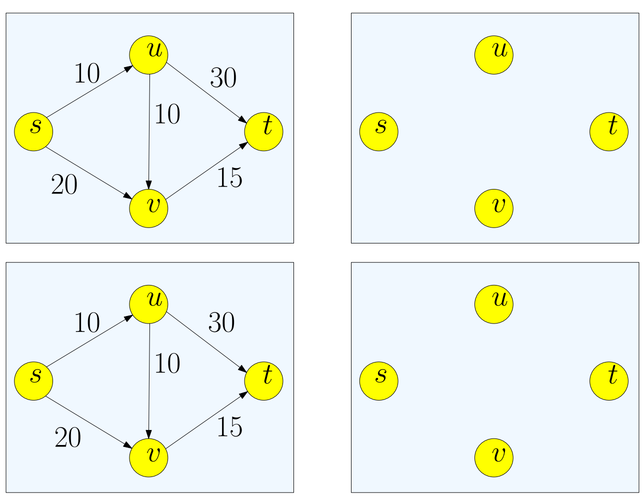
17.2.0.8 Example



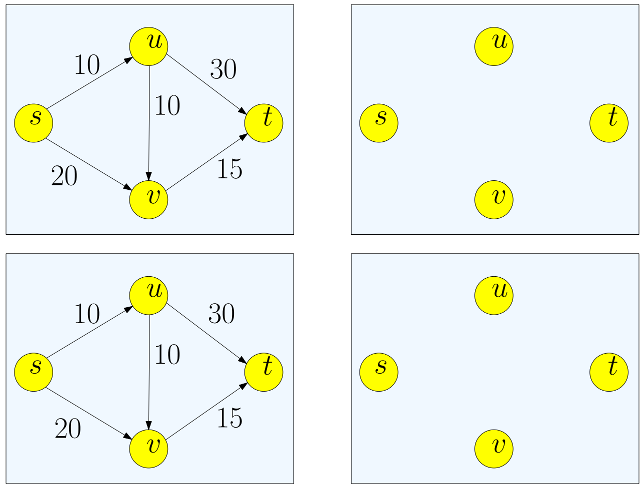
17.2.0.9 Example continued



17.2.0.10 Example continued



17.2.0.11 Example continued



17.3 Correctness and Analysis

17.3.1 Termination

17.3.1.1 Properties about Augmentation: Flow

Lemma 17.3.1 *If f is a flow and P is a simple s-t path in G_f, then $f' = \texttt{augment}(f, P)$ is also a flow.*

Proof: Verify that f' is a flow. Let b be augmentation amount.

- *Capacity constraint:* If $(u, v) \in P$ is a forward edge then $f'(e) = f(e) + b$ and $b \leq c(e) - f(e)$. If $(u, v) \in P$ is a backward edge, then letting $e = (v, u)$, $f'(e) = f(e) - b$ and $b \leq f(e)$. Both cases $0 \leq f'(e) \leq c(e)$.

- *Conservation constraint:* Let v be an internal node. Let e_1, e_2 be edges of P incident to v. Four cases based on whether e_1, e_2 are forward or backward edges. Check cases (see fig next slide).

$\blacksquare$

17.3.1.2 Properties about Augmentation: Conservation Constraint

17.3.1.3 Properties about Augmentation: Integer Flow

Lemma 17.3.2 *At every stage of the Ford-Fulkerson algorithm, the flow values $f(e)$ and the residual capacities in G_f are integers*

Proof: Initial flow and residual capacities are integers. Suppose lemma holds for j iterations. Then in $(j + 1)$st iteration, minimum capacity edge b is an integer, and so flow after augmentation is an integer. $\blacksquare$

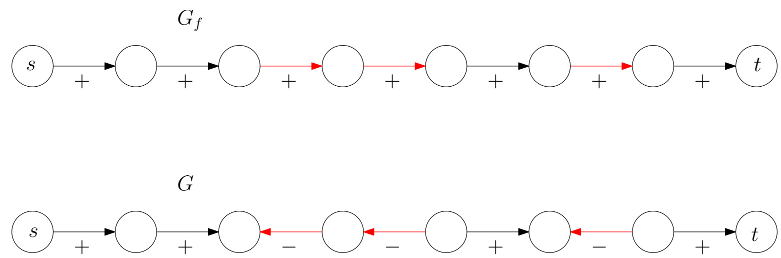


Figure 17.3: Augmenting path P in G_f and corresponding change of flow in G. Red edges are backward edges.

17.3.1.4 Progress in Ford-Fulkerson

Proposition 17.3.3 *Let f be a flow and f' be flow after one augmentation. Then $v(f) < v(f')$.*

Proof: Let P be an augmenting path, i.e., P is a simple s-t path in residual graph

- First edge e in P must leave s
- Original network G has no incoming edges to s; hence e is a forward edge
- P is simple and so never returns to s
- Thus, value of flow increases by the flow on edge e

■

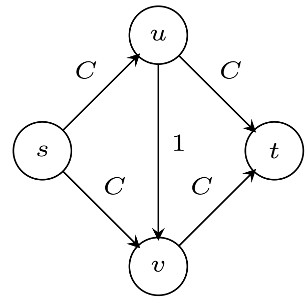
17.3.1.5 Termination Proof

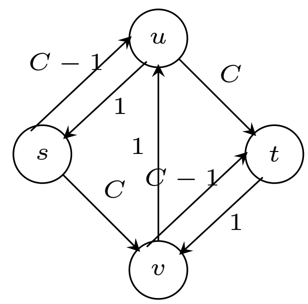
Theorem 17.3.4 *Let C be the minimum cut value; in particular $C \leq \sum_{e \text{ out of } s} c(e)$. Ford-Fulkerson algorithm terminates after finding at most C augmenting paths.*

Proof: The value of the flow increases by at least 1 after each augmentation. Maximum value of flow is at most C. ■

Running time

- Number of iterations $\leq C$
- Number of edges in $G_f \leq 2m$
- Time to find augmenting path is $O(n + m)$
- Running time is $O(C(n + m))$ (or $O(mC)$).





17.3.1.6 Efficiency of Ford-Fulkerson

Running time $= O(mC)$ is not polynomial. Can the running time be as $\Omega(mC)$ or is our analysis weak? Ford-Fulkerson can take $\Omega(C)$ iterations.

17.3.2 Correctness

17.3.2.1 Correctness of Ford-Fulkerson Augmenting Path Algorithm

Question: When the algorithm terminates, is the flow computed the maximum s-t flow?

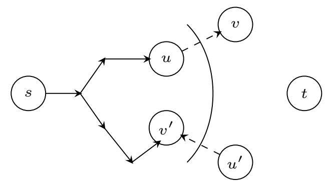
Proof idea: show a cut of value equal to the flow. Also shows that maximum flow is equal to minimum cut!

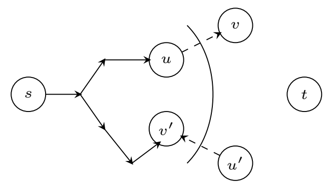
17.3.2.2 Recalling Cuts

Definition 17.3.5 *Given a flow network an s-t cut is a set of edges $E' \subset E$ such that removing E'* disconnects *s from t: in other words there is no directed $s \to t$ path in $E - E'$.* Capacity *of cut E' is $\sum_{e \in E'} c(e)$.*

Let $A \subset V$ such that

- $s \in A$, $t \notin A$
- $B = V - A$ and hence $t \in B$





Define $(A, B) = \{(u, v) \in E \mid u \in A, v \in B\}$

Claim 17.3.6 *(A, B) is an s-t cut.*

Recall: Every *minimal* s-t cut E' is a cut of the form (A, B).

17.3.2.3 Ford-Fulkerson Correctness

Lemma 17.3.7 *If there is no s-t path in G_f then there is some cut (A, B) such that $v(f) = c(A, B)$*

Proof: Let A be all vertices reachable from s in G_f; $B = V \setminus A$

- $s \in A$ and $t \in B$. So (A, B) is an s-t cut in G
- If $e = (u, v) \in G$ with $u \in A$ and $v \in B$, then $f(e) = c(e)$ (*saturated* edge) because otherwise v is reachable from s in G_f

∎

17.3.2.4 Lemma Proof Continued

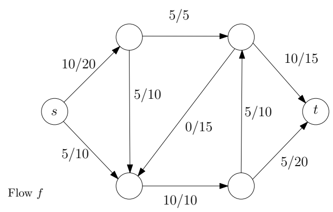
Proof:

- If $e = (u', v') \in G$ with $u' \in B$ and $v' \in A$, then $f(e) = 0$ because otherwise u' is reachable from s in G_f
- Thus,

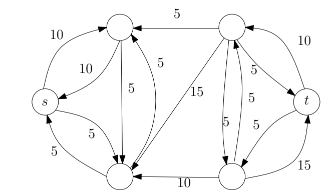
$$
\begin{aligned}
v(f) &= f^{\text{out}}(A) - f^{\text{in}}(A) \\
&= f^{\text{out}}(A) - 0 \\
&= c(A, B) - 0 \\
&= c(A, B)
\end{aligned}
$$

∎

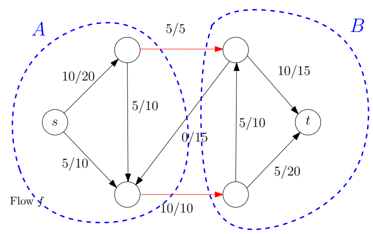
17.3.2.5 Example



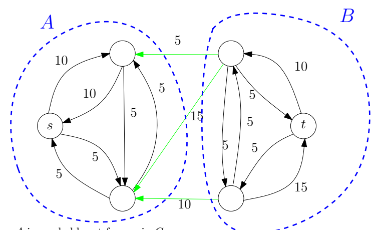
Flow f



Residual graph G_f: no s-t path



Flow f



A is reachable set from s in G_f

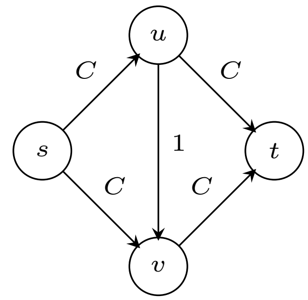
17.3.2.6 Ford-Fulkerson Correctness

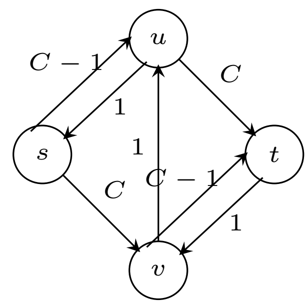
Theorem 17.3.8 *The flow returned by the algorithm is the maximum flow.*

Proof:

- For any flow f and s-t cut (A, B), $v(f) \leq c(A, B)$
- For flow f^* returned by algorithm, $v(f^*) = c(A^*, B^*)$ for some s-T cut (A^*, B^*)
- Hence, f^* is maximum

$\blacksquare$





17.3.2.7 Max-Flow Min-Cut Theorem and Integrality of Flows

Theorem 17.3.9 *For any network G, the value of a maximum s-t flow is equal to the capacity of the minimum s-t cut.*

Proof: Ford-Fulkerson algorithm terminates with a maximum flow of value equal to the capacity of a (minimum) cut. ■

17.3.2.8 Max-Flow Min-Cut Theorem and Integrality of Flows

Theorem 17.3.10 *For any network G with integer capacities, there is a maximum s-t flow that is integer valued.*

Proof: Ford-Fulkerson algorithm produces an integer valued flow when capacities are integers. ■

17.4 Polynomial Time Algorithms

17.4.0.9 Efficiency of Ford-Fulkerson

Running time $= O(mC)$ is not polynomial. Can the upper bound be achieved?

17.4.0.10 Polynomial Time Algorithms

Question: Is there a polynomial time algorithm for maxflow?

Question: Is there a variant of Ford-Fulkerson that leads to a polynomial time algorithm? Can we choose an augmenting path in some clever way? Yes! Two variants.

- Choose the augmenting path with largest bottleneck capacity.
- Choose the shortest augmenting path.

17.4.1 Capacity Scaling Algorithm

17.4.1.1 Augmenting Paths with Large Bottleneck Capacity

- Pick augmenting paths with largest bottleneck capacity in each iteration of Ford-Fulkerson
- How do we find path with largest bottleneck capacity?
  - Assume we know Δ the bottleneck capacity
  - Remove all edges with residual capacity $\leq \Delta$
  - Check if there is a path from s to t
  - Do binary search to find largest Δ
  - Running time: $O(m \log C)$
- Can we bound the number of augmentations? Can show that in $O(m \log C)$ augmentations the algorithm reaches a max flow. This leads to an $O(m^2 \log^2 C)$ time algorithm.

17.4.1.2 Augmenting Paths with Large Bottleneck Capacity

How do we find path with largest bottleneck capacity?

- Max bottleneck capacity is one of the edge capacities. Why?
- Can do binary search on the edge capacities. First, sort the edges by their capacities and then do binary search on that array as before.
- Algorithm's running time is $O(m \log m)$.
- Different algorithm that also leads to $O(m \log m)$ time algorithm by adapting Prim's algorithm.