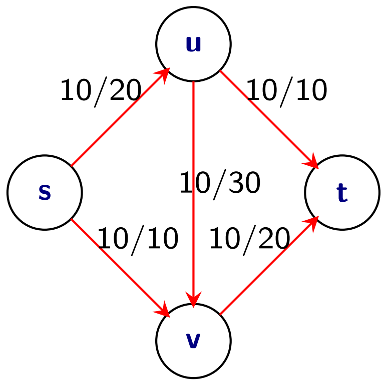
Part I

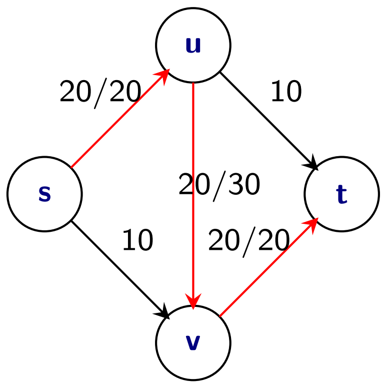
Algorithm(s) for Maximum Flow

Greedy Approach



1. Begin with $f(e) = 0$ for each edge
2. Find a s-t path P with $f(e) < c(e)$ for every edge $e \in P$
3. *Augment* flow along this path
4. Repeat augmentation for as long as possible.

Greedy Approach: Issues



1. Begin with $\mathbf{f(e) = 0}$ for each edge
2. Find a $\mathbf{s}$-$\mathbf{t}$ path $\mathbf{P}$ with $\mathbf{f(e) < c(e)}$ for every edge $\mathbf{e \in P}$
3. Augment flow along this path
4. Repeat augmentation for as long as possible.

Greedy can get stuck in sub-optimal flow!
Need to "push-back" flow along edge $\mathbf{(u, v)}$

Residual Graph

Definition

For a network $\mathbf{G = (V, E)}$ and flow $\mathbf{f}$, the residual graph $\mathbf{G_f = (V', E')}$ of $\mathbf{G}$ with respect to $\mathbf{f}$ is

- $\mathbf{V' = V}$
- Forward Edges: For each edge $\mathbf{e \in E}$ with $\mathbf{f(e) < c(e)}$, we $\mathbf{e \in E'}$ with capacity $\mathbf{c(e) - f(e)}$
- Backward Edges: For each edge $\mathbf{e = (u, v) \in E}$ with $\mathbf{f(e) > 0}$, we $\mathbf{(v, u) \in E'}$ with capacity $\mathbf{f(e)}$

Residual Graph Example

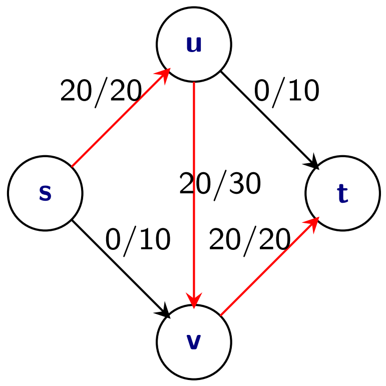


Figure: Flow in red edges

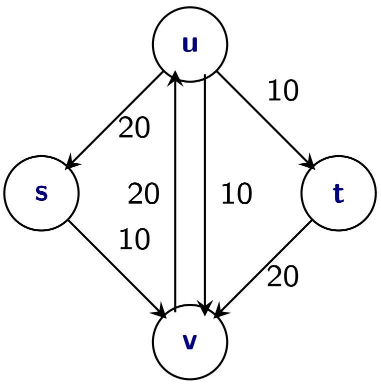


Figure: Residual Graph

Residual Graph Property

Observation: Residual graph captures the "residual" problem exactly.

Lemma

Let $\mathbf{f}$ *be a flow in* $\mathbf{G}$ *and* $\mathbf{G_f}$ *be the residual graph. If* $\mathbf{f'}$ *is a flow in* $\mathbf{G_f}$ *then* $\mathbf{f} + \mathbf{f'}$ *is a flow in* $\mathbf{G}$ *of value* $\mathbf{v(f)} + \mathbf{v(f')}$.

Lemma

Let $\mathbf{f}$ *and* $\mathbf{f'}$ *be two flows in* $\mathbf{G}$ *with* $\mathbf{v(f')} \geq \mathbf{v(f)}$. *Then there is a flow* $\mathbf{f''}$ *of value* $\mathbf{v(f')} - \mathbf{v(f)}$ *in* $\mathbf{G_f}$.

Definition of + and - for flows is intuitive and the above lemmas are easy in some sense but a bit messy to formally prove.

Residual Graph Property: Implication

Recursive algorithm for finding a maximum flow:

MaxFlow($\mathbf{G}, \mathbf{s}, \mathbf{t}$):
    If the flow from $\mathbf{s}$ to $\mathbf{t}$ is $\mathbf{0}$
        return $\mathbf{0}$
    Find any flow $\mathbf{f}$ with $\mathbf{v(f)} > \mathbf{0}$ in $\mathbf{G}$
    Recursively compute a maximum flow $\mathbf{f'}$ in $\mathbf{G_f}$
    Output the flow $\mathbf{f} + \mathbf{f'}$

Iterative algorithm for finding a maximum flow:

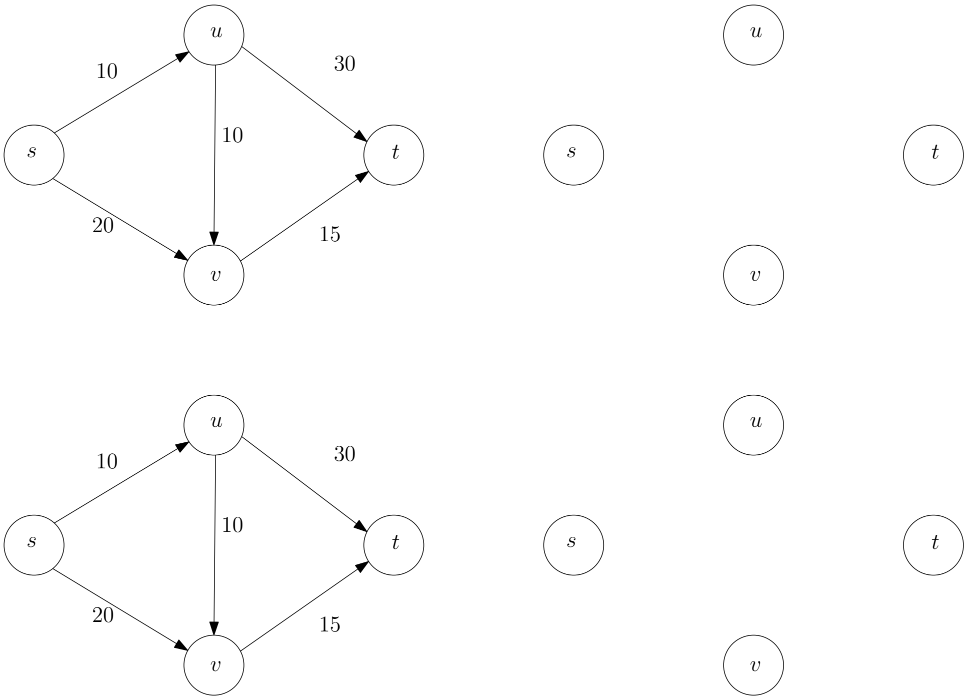
MaxFlow($\mathbf{G}, \mathbf{s}, \mathbf{t}$):
    Start with flow $\mathbf{f}$ that is $\mathbf{0}$ on all edges
    While there is a flow $\mathbf{f'}$ in $\mathbf{G_f}$ with $\mathbf{v(f')} > \mathbf{0}$ do
        $\mathbf{f}$ = $\mathbf{f} + \mathbf{f'}$
        Update $\mathbf{G_f}$
    endWhile
    Output $\mathbf{f}$

Ford-Fulkerson Algorithm

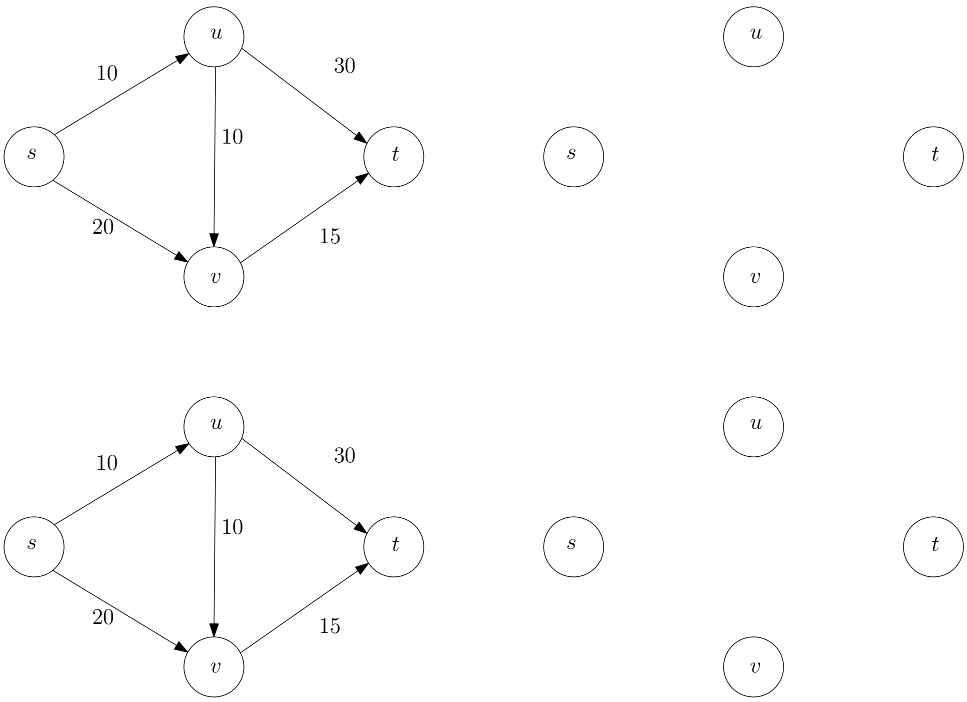
```
for every edge e, f(e) = 0
G_f is residual graph of G with respect to f
while G_f has a simple s-t path
    let P be simple s-t path in G_f
    f = augment(f,P)
    Construct new residual graph G_f

augment(f,P)
    let b be bottleneck capacity, i.e., min capacity of edges in P
    for each edge (u,v) in P
        if e=(u,v) is a forward edge
            f(e) = f(e) + b
        else (* (u,v) is a backward edge *)
            let e = (v,u) (* (v,u) is in G *)
            f(e) = f(e) - b
    return f
```

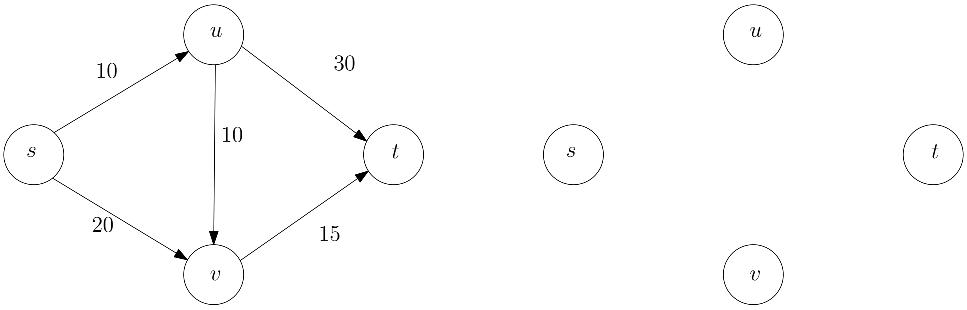
Example

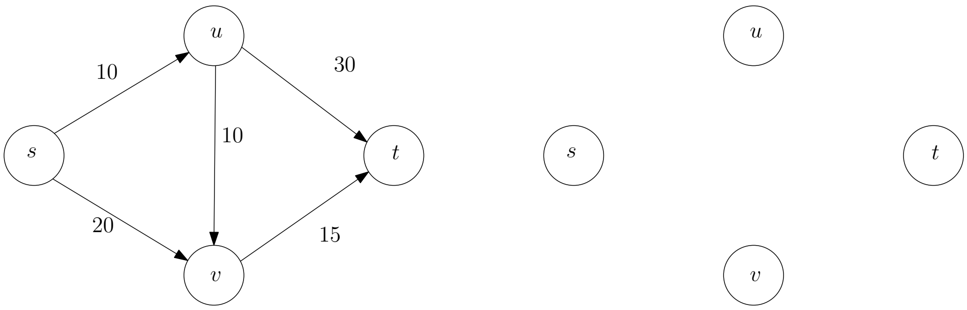


Example continued

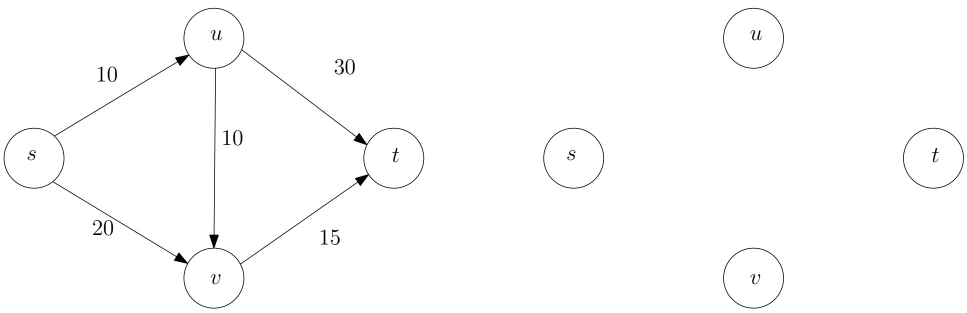


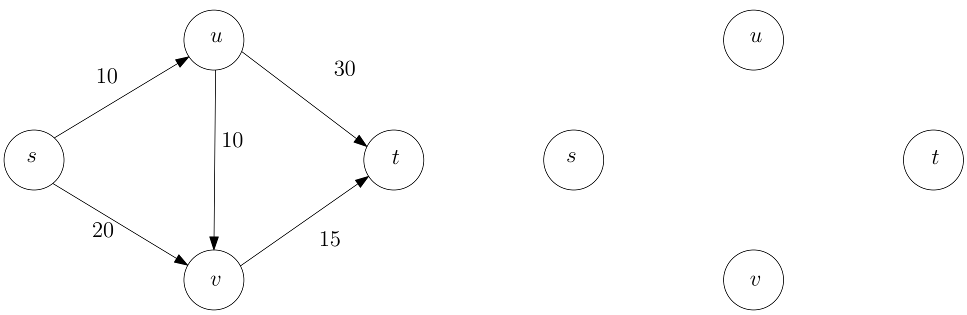
Example continued





Example continued





Properties about Augmentation: Flow

Lemma

If $\mathbf{f}$ *is a flow and* $\mathbf{P}$ *is a simple* $\mathbf{s}$-$\mathbf{t}$ *path in* $\mathbf{G_f}$*, then* $\mathbf{f'} = \mathtt{augment}(\mathbf{f}, \mathbf{P})$ *is also a flow.*

Proof.

Verify that $\mathbf{f'}$ is a flow. Let $\mathbf{b}$ be augmentation amount.

- Capacity constraint: If $\mathbf{(u, v) \in P}$ is a forward edge then $\mathbf{f'(e) = f(e) + b}$ and $\mathbf{b \leq c(e) - f(e)}$. If $\mathbf{(u, v) \in P}$ is a backward edge, then letting $\mathbf{e = (v, u)}$, $\mathbf{f'(e) = f(e) - b}$ and $\mathbf{b \leq f(e)}$. Both cases $\mathbf{0 \leq f'(e) \leq c(e)}$.
- Conservation constraint: Let $\mathbf{v}$ be an internal node. Let $\mathbf{e_1, e_2}$ be edges of $\mathbf{P}$ incident to $\mathbf{v}$. Four cases based on whether $\mathbf{e_1, e_2}$ are forward or backward edges. Check cases (see fig next slide). □

Properties about Augmentation: Conservation Constraint

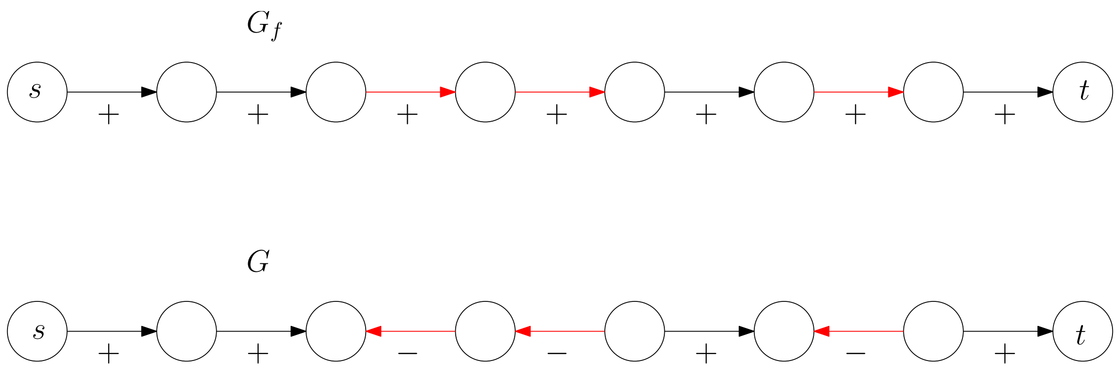


Figure: Augmenting path $\mathbf{P}$ in $\mathbf{G_f}$ and corresponding change of flow in $\mathbf{G}$. Red edges are backward edges.

Properties about Augmentation: Integer Flow

Lemma

At every stage of the Ford-Fulkerson algorithm, the flow values $\mathbf{f(e)}$ and the residual capacities in $\mathbf{G_f}$ are integers

Proof.

Initial flow and residual capacities are integers. Suppose lemma holds for $\mathbf{j}$ iterations. Then in $\mathbf{j+1}$st iteration, minimum capacity edge $\mathbf{b}$ is an integer, and so flow after augmentation is an integer. □

Progress in Ford-Fulkerson

Proposition

Let $\mathbf{f}$ be a flow and $\mathbf{f'}$ be flow after one augmentation. Then $\mathbf{v(f) < v(f')}$.

Proof.

Let $\mathbf{P}$ be an augmenting path, i.e., $\mathbf{P}$ is a simple $\mathbf{s}$-$\mathbf{t}$ path in residual graph

- First edge $\mathbf{e}$ in $\mathbf{P}$ must leave $\mathbf{s}$
- Original network $\mathbf{G}$ has no incoming edges to $\mathbf{s}$; hence $\mathbf{e}$ is a forward edge
- $\mathbf{P}$ is simple and so never returns to $\mathbf{s}$
- Thus, value of flow increases by the flow on edge $\mathbf{e}$ □

Termination Proof

Theorem

Let $\mathbf{C}$ *be the minimum cut value; in particular* $\mathbf{C} \leq \sum_{\mathbf{e} \text{ out of } \mathbf{s}} \mathbf{c(e)}$. *Ford-Fulkerson algorithm terminates after finding at most* $\mathbf{C}$ *augmenting paths.*

Proof.

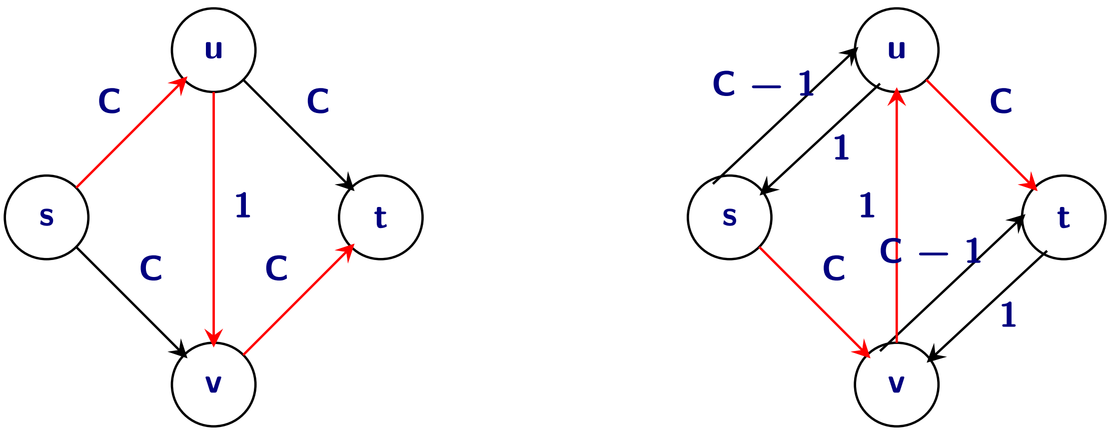
The value of the flow increases by at least 1 after each augmentation. Maximum value of flow is at most $\mathbf{C}$. □

Running time

- Number of iterations $\leq \mathbf{C}$
- Number of edges in $\mathbf{G_f} \leq \mathbf{2m}$
- Time to find augmenting path is $\mathbf{O(n + m)}$
- Running time is $\mathbf{O(C(n + m))}$ (or $\mathbf{O(mC)}$).

Efficiency of Ford-Fulkerson

Running time $= \mathbf{O(mC)}$ is not polynomial. Can the running time be as $\mathbf{\Omega(mC)}$ or is our analysis weak?



Ford-Fulkerson can take $\mathbf{\Omega(C)}$ iterations.

Correctness of Ford-Fulkerson Augmenting Path Algorithm

Question: When the algorithm terminates, is the flow computed the maximum $\mathbf{s}$-$\mathbf{t}$ flow?

Proof idea: show a cut of value equal to the flow. Also shows that maximum flow is equal to minimum cut!

Recalling Cuts

Definition

Given a flow network an $\mathbf{s}$-$\mathbf{t}$ cut is a set of edges $\mathbf{E}' \subset \mathbf{E}$ such that removing $\mathbf{E}'$ *disconnects* $\mathbf{s}$ from $\mathbf{t}$: in other words there is no directed $\mathbf{s} \to \mathbf{t}$ path in $\mathbf{E} - \mathbf{E}'$. *Capacity* of cut $\mathbf{E}'$ is $\sum_{\mathbf{e} \in \mathbf{E}'} \mathbf{c}(\mathbf{e})$.

Let $\mathbf{A} \subset \mathbf{V}$ such that

- $\mathbf{s} \in \mathbf{A}$, $\mathbf{t} \not\in \mathbf{A}$
- $\mathbf{B} = \mathbf{V} - \mathbf{A}$ and hence $\mathbf{t} \in \mathbf{B}$

Define $(\mathbf{A}, \mathbf{B}) = \{(\mathbf{u}, \mathbf{v}) \in \mathbf{E} \mid \mathbf{u} \in \mathbf{A}, \mathbf{v} \in \mathbf{B}\}$

Claim

$(\mathbf{A}, \mathbf{B})$ *is an* $\mathbf{s}$-$\mathbf{t}$ *cut.*

Recall: Every *minimal* $\mathbf{s}$-$\mathbf{t}$ cut $\mathbf{E}'$ is a cut of the form $(\mathbf{A}, \mathbf{B})$.

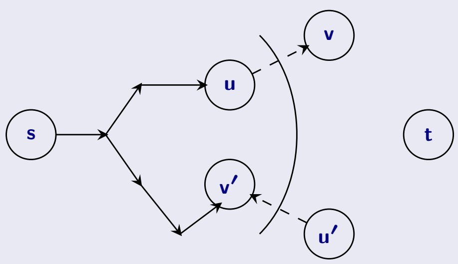
Ford-Fulkerson Correctness

Lemma

If there is no s-t *path in* G_f *then there is some cut* (A, B) *such that* $v(f) = c(A, B)$

Proof.

Let A be all vertices reachable from s in G_f; $B = V \setminus A$

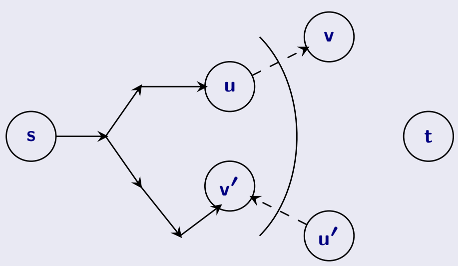


- $s \in A$ and $t \in B$. So (A, B) is an s-t cut in G
- If $e = (u, v) \in G$ with $u \in A$ and $v \in B$, then $f(e) = c(e)$ (saturated edge) because otherwise v is reachable from s in G_f

□

Lemma Proof Continued

Proof.

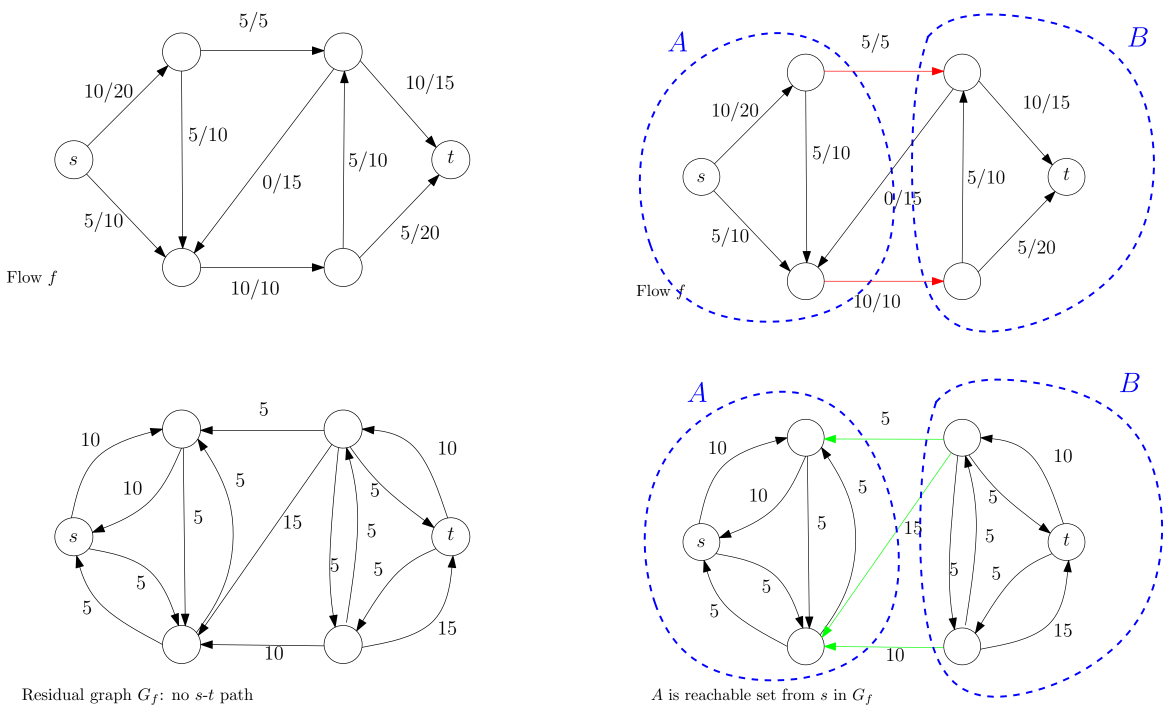


- If $e = (u', v') \in G$ with $u' \in B$ and $v' \in A$, then $f(e) = 0$ because otherwise u' is reachable from s in G_f
- Thus,

$$\begin{aligned} v(f) &= f^{out}(A) - f^{in}(A) \\ &= f^{out}(A) - 0 \\ &= c(A, B) - 0 \\ &= c(A, B) \end{aligned}$$

□

Example



Ford-Fulkerson Correctness

Theorem

The flow returned by the algorithm is the maximum flow.

Proof.

- For any flow $\mathbf{f}$ and $\mathbf{s}$-$\mathbf{t}$ cut $\mathbf{(A, B)}$, $\mathbf{v(f) \leq c(A, B)}$
- For flow $\mathbf{f^*}$ returned by algorithm, $\mathbf{v(f^*) = c(A^*, B^*)}$ for some $\mathbf{s}$-$\mathbf{T}$ cut $\mathbf{(A^*, B^*)}$
- Hence, $\mathbf{f^*}$ is maximum

$\square$

Max-Flow Min-Cut Theorem and Integrality of Flows

Theorem

For any network $\mathbf{G}$*, the value of a maximum* $\mathbf{s}$-$\mathbf{t}$ *flow is equal to the capacity of the minimum* $\mathbf{s}$-$\mathbf{t}$ *cut.*

Proof.

Ford-Fulkerson algorithm terminates with a maximum flow of value equal to the capacity of a (minimum) cut. □

Theorem

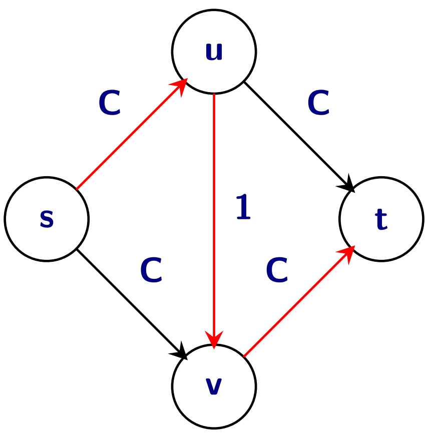
For any network $\mathbf{G}$ *with integer capacities, there is a maximum* $\mathbf{s}$-$\mathbf{t}$ *flow that is integer valued.*

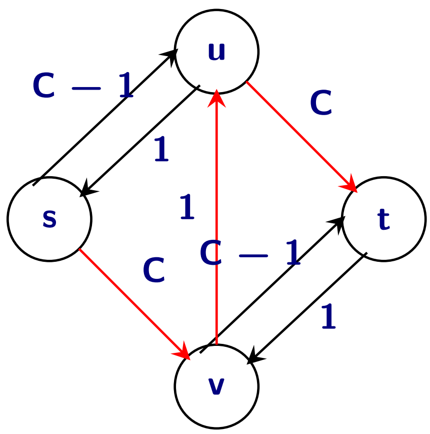
Proof.

Ford-Fulkerson algorithm produces an integer valued flow when capacities are integers. □

Efficiency of Ford-Fulkerson

Running time $= \mathbf{O(mC)}$ is not polynomial. Can the upper bound be achieved?





Polynomial Time Algorithms

Question: Is there a polynomial time algorithm for maxflow?

Question: Is there a variant of Ford-Fulkerson that leads to a polynomial time algorithm? Can we choose an augmenting path in some clever way? Yes! Two variants.

- Choose the augmenting path with largest bottleneck capacity.
- Choose the shortest augmenting path.

Augmenting Paths with Large Bottleneck Capacity

- Pick augmenting paths with largest bottleneck capacity in each iteration of Ford-Fulkerson
- How do we find path with largest bottleneck capacity?
  - Assume we know Δ the bottleneck capacity
  - Remove all edges with residual capacity $\leq \Delta$
  - Check if there is a path from s to t
  - Do binary search to find largest Δ
  - Running time: $O(m \log C)$
- Can we bound the number of augmentations? Can show that in $O(m \log C)$ augmentations the algorithm reaches a max flow. This leads to an $O(m^2 \log^2 C)$ time algorithm.

Book gives a simpler variant called Capacity Scaling algorithm that runs in $O(m^2 \log C)$ time.

Augmenting Paths with Large Bottleneck Capacity

How do we find path with largest bottleneck capacity?

- Max bottleneck capacity is one of the edge capacities. Why?
- Can do binary search on the edge capacities. First, sort the edges by their capacities and then do binary search on that array as before.
- Algorithm's running time is $O(m \log m)$.
- Different algorithm that also leads to $O(m \log m)$ time algorithm by adapting Prim's algorithm.

Removing Dependence on C

- [Edmonds-Karp, Dinitz] Picking augmenting paths with fewest number of edges yields a $O(m^2 n)$ algorithm, i.e., independent of C. Such an algorithm is called a strongly polynomial time algorithm since the running time does not depend on the numbers (assuming RAM model). (Many implementation of Ford-Fulkerson would actually use shortest augmenting path if they use BFS to find an s-t path).
- Further improvements can yield algorithms running in $O(mn \log n)$, or $O(n^3)$.

Finding a Minimum Cut

Question: How do we find an actual minimum $\mathbf{s}$-$\mathbf{t}$ cut?
Proof gives the algorithm!

- Compute an $\mathbf{s}$-$\mathbf{t}$ maximum flow $\mathbf{f}$ in $\mathbf{G}$
- Obtain the residual graph $\mathbf{G_f}$
- Find the nodes $\mathbf{A}$ reachable from $\mathbf{s}$ in $\mathbf{G_f}$
- Output the cut $(\mathbf{A}, \mathbf{B}) = \{(\mathbf{u}, \mathbf{v}) \mid \mathbf{u} \in \mathbf{A}, \mathbf{v} \in \mathbf{B}\}$. Note: The cut is found in $\mathbf{G}$ while $\mathbf{A}$ is found in $\mathbf{G_f}$

Running time is essentially the same as finding a maximum flow.

Note: Given $\mathbf{G}$ and a flow $\mathbf{f}$ there is a linear time algorithm to check if $\mathbf{f}$ is a maximum flow and if it is, outputs a minimum cut. How?